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A constructive approach to Markov compacta

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Introduction

In this paper we present a new approach (which we call *constructive*) to a class of topological spaces, named in the papers [Dra06], [Paw15] as *Markov compacta*. By definition a Markov compactum is the limit of an inverse sequence of finite simplicial complexes of a quite special kind. The notion of such an object is important in geometric group theory, as Gromov boundaries of hyperbolic groups, and more generally ideal boundaries of various groups can be described as Markov compacta [Paw15].

The definitions of Dranishnikov and Pawlik are nonconstructive in the sense that they describe what conditions a given inverse system must meet for its limit to become a Markov compactum. The motivation for our approach came from the observation that Markov compacta from some class described in [Paw15] are *finitely describable*, which means that each space in this class is uniquely determined by a finite set of data (via a certain algorithmic procedure). This aspect of Markov compacta was not addressed in a satisfactory way in the existing literature. We present a way in which certain specific collections of finite data induce inverse sequences of spaces via certain recursive procedures. Moreover, as a corollary, we show that Gromov boundaries of hyperbolic groups can be described as inverse limit of a special kind of these inverse sequences. We call the inverse limit spaces of the induced inverse sequences constructive Markov compacta.

0 Preliminaries

In this section we recall the definition of a simplicial complex, which will be used throughout the paper. The definition is based on the ideas from Chapter 2 of [Hat02].

Definition 0.1. A *standard i-dimensional simplex* is the set

$$\Delta^i = \{(t_0, \ldots, t_i) \in \mathbb{R}^{i+1} : \sum_j t_j = 1 \text{ and } t_j \geq 0 \text{ for all j}\}$$

The *vertices* of Δ^i are the unit coordinate vectors $e_j \in \mathbf{R}^{i+1}, 0 \leq j \leq i$. A *face* of Δ^i is the affine convex span of any nonempty subset of the vertex set of Δ^i. A face is called *proper* if the corresponding vertex set is a proper subset of the full vertex set. The *boundary* $\partial\Delta^i$ is the union of all proper faces of Δ^i, and the *interior* $\mathrm{int}\Delta^i$ is $\Delta^i \setminus \partial\Delta^i$. In particular, we have $\partial\Delta^0 = \emptyset$ and $\mathrm{int}\Delta^0 = \Delta^0$. The faces of Δ^i are also viewed as standard simplices of the corresponding dimension, via obvious identifications with the corresponding Δ^j's.

Definition 0.2. *A geometric i-dimensional simplex* (*or a geometric i-simplex*) in a topological space X is a subspace $\sigma \subset X$ together with a homeomorphism $\phi_\sigma : \Delta^i \to \sigma$ from the standard i-simplex. (We do not distinguish homeomorphisms that differ by precomposition with any affine isomorphism of Δ^i.) The homeomorphism ϕ_σ above is called the *characteristic map* of the simplex σ. A *face* of a geometric simplex is the image $\tau = \phi_\sigma(\Delta^j)$ through ϕ_σ of any face $\Delta^j \subset \Delta^i$, where $j \leq i$, together with the restricted homeomorphism $\phi_{\sigma\restriction\Delta^j} : \Delta^j \to \tau$. A face τ of a geometric simplex σ as above is *proper* if the corresponding simplex Δ^j is a proper face of Δ^i. The *boundary* $\partial\sigma$ of σ is the image through ϕ_σ of the boundary of the corresponding Δ^i, and the *interior* $\mathrm{int}(\sigma)$ of a geometric simplex σ is the image through ϕ_σ of the interior of the corresponding Δ^i.

Definition 0.3. A *simplicial complex* is a space X equipped with a distinguished family $S(X)$ of geometric simplices in X such that

1. $S(X)$ is closed under taking faces,
2. each point of X belongs to the interior of precisely one simplex from $S(X)$,
3. the intersection of any two simplices from $S(X)$ is either empty or a face in each of the simplices,
4. for each simplex σ of X the union of the interiors of all simplices of X that contain σ (called the *open star* of σ with respect to X) is an open subset of X.

The family $S(X)$ is called *the set of simplices of X*. A geometric i-dimensional simplex from $S(X)$ is an *i-dimensional simplex of X*, or shortly an *i-simplex.* A *vertex* of X is any point $p \in X$ such that the singleton $\{p\}$ is a 0-simplex of X. Note that it follows from condition 3 in the above definition that every simplex of X, as an element of $S(X)$, is uniquely determined by the set of its vertices. We thus say of any simplex that it is the simplex *spanned* on the set of its vertices.

A simplicial complex X is *finite* if the corresponding set $S(X)$ of its simplices is finite. X is *finite-dimensional* if there is a universal bound from above for the dimension of its simplices.

Definition 0.4. Let X be a simplicial complex and let v be a vertex of X. The *closed star of v with respect to X*, denoted by $\mathrm{st}(v, X)$ is the closure of the open star of v with respect to X. In other words, it is the union of all simplices of X that contain v.

Definition 0.5. Let X and Y be simplicial complexes. A continuous map $f : X \to Y$ is *simplicial* if for any simplex $\sigma \subset X$ there is a simplex $\tau \subset Y$ such that the restriction $f_{\restriction\sigma}$ is an affine map onto τ. More precisely, the latter means that if $\phi_\sigma : \Delta^{\dim\sigma} \to \sigma$ and $\phi_\tau : \Delta^{\dim\tau} \to \tau$ are the characteristic maps of σ and τ, then the composition $\phi_\tau^{-1} \circ f \circ \phi_\sigma : \Delta^{\dim\sigma} \to \Delta^{\dim\tau}$ is a surjective affine map that sends vertices to vertices. A *simplicial isomorphism* is a bijection f such that both f and f^{-1} are simplicial maps.

1 Assembly system and semi-barycentric maps

In this section we describe general concepts and constructions used in the paper.

1.1 Assembly systems

Definition 1.1. Let K be a simplicial complex. Assume that

- with every simplex $\tau \subset K$ there is associated a topological space Y_τ,
- with every pair $\rho \subset \tau$ of simplices from K there is an associated embedding $i_{\rho\tau} : Y_\rho \to Y_\tau$.

We also assume that maps from the family $\{i_{\rho\tau}\}_{\rho\subset\tau}$ satisfy the following condition: for simplices ρ, τ, ν from the complex K such that $\rho \subset \tau \subset \nu$ we have $i_{\rho\nu} = i_{\tau\nu} \circ i_{\rho\tau}$ (convention: $i_{\rho\rho} = id_{Y_\rho}$). Call this the *composition property.* We also assume that for simplices τ_1, τ_2 and σ from the complex K such that $\tau_1, \tau_2 \subset \sigma$, we have $i_{\tau_1\sigma}(Y_{\tau_1}) \cap i_{\tau_2\sigma}(Y_{\tau_2}) = i_{\tau_1\cap\tau_2,\sigma}(Y_{\tau_1\cap\tau_2})$ (convention: $i_{\emptyset\sigma}(Y_\emptyset) = \emptyset$). We call this the *intersection property.* A system $\mathcal{A} = \big(\{Y_\tau\}_{\tau\in S(K)}, \{i_{\rho\tau}\}_{\rho\subset\tau}\big)$ which satisfies the above conditions is called an *assembly system* over the complex K.
Whenever it won't lead to ambiguities, instead of writing "let $\mathcal{A} = \big(\{Y_\tau\}_{\tau\in S(K)}, \{i_{\rho\tau}\}_{\rho\subset\tau}\big)$ be an assembly system" we will simply write "let $\mathcal{A}$ be an assembly system" assuming that there is given a collection of topological spaces $\{Y_\tau\}_{\tau\in S(K)}$ and embeddings $\{i_{\rho\tau}\}_{\rho\subset\tau}$ satisfying the composition and intersection properties given in the definition above.

Definition 1.2. Let $\mathcal{A}$ be an assembly system over a simplicial complex K. The $\mathcal{A}$*-quotient* is the topological space

$$Y = \coprod_{\tau\subset K} Y_\tau \big/ \sim \tag{1}$$

where $\sim$ is the smallest equivalence relation generated by the relation $\approx$:

$$p \approx q \iff p \in Y_\rho, q \in Y_\tau, \rho \neq \tau, \rho \subset \tau \text{ and } i_{\rho\tau}(p) = q \tag{2}$$

The relation $\sim$ is effectively characterised in the following lemma.

Lemma 1.3. *Let $\mathcal{A}$ be an assembly system over a simplicial complex K and let $\sim$ be the equivalence relation from Definition 1.2. Then for simplices $\rho, \tau \subset K$ and points $p \in Y_\tau, q \in Y_\rho$ we have $p \sim q \iff \tau \cap \rho \neq \emptyset$ and there exists a point $r \in Y_{\tau\cap\rho}$ such that $i_{\tau\cap\rho,\tau}(r) = p$ and $i_{\tau\cap\rho,\rho}(r) = q$.*

Proof. The proof is a fairly easy computation, which involves only using the assumption that the maps from the family $\{i_{\rho\tau}\}_{\rho\subset\tau}$ are injective and that they satisfy the composition and intersection property, and we skip the details. □

Comment. In the setting as in the statement of Lemma 1.3 we say that the point r *glues together* the points p and q.

Fact 1.4. *Let $\mathcal{A}$ be an assembly system over a simplicial complex K and let Y be the $\mathcal{A}$-quotient. The restriction of the quotient map $\pi : \coprod_{\tau\subset K} Y_\tau \to Y$ to each of the spaces Y_τ is one-to-one.*

Proof. Let $p, q \in Y_\tau$ for some $\tau \subset K$. If $\pi(p) = \pi(q)$, it follows directly from Lemma 1.3 that $p = q$. □

Remark. The above fact lets us adopt the following convention: in the course of this paper we will identify the spaces Y_τ with the corresponding subspaces of Y, and points of Y_τ with their images in Y.

For the purposes of this paper, we will now describe a simplicial version of assembly systems.

Definition 1.5. A *simplicial assembly system* over a simplicial complex K is an assembly system $\mathcal{A}$, in which $\{Y_\tau\}_{\tau\in S(K)}$ is a family of simplicial complexes and $\{i_{\rho\tau}\}_{\rho\subset\tau}$ is a family of simplicial embeddings.

Remark. We will adopt the following convention: whenever we speak of a simplicial assembly system, we denote it by A, dropping the calligraphic font.

It is natural to consider a simplicial version of Lemma 1.3, which is slightly stronger.

Lemma 1.6. *Let A be a simplicial assembly system. If for some points $z_j \in int(\sigma_j) \subset Y_{\tau_j}, j = 1, 2$ we have $z_1 \sim z_2$, then, in addition to the assertions of Lemma 1.3, there exists a simplex $\sigma \subset Y_{\tau_1\cap\tau_2}$ such that $i_{\tau_1\cap\tau_2,\tau_j}(\sigma) = \sigma_j$ for $j = 1, 2$. Moreover, the equality $\pi(\sigma_1) = \pi(\sigma_2)$ holds as well.*

Proof. In the course of the proof we will use the following facts:

- in a simplicial complex every point lies in the interior of exactly one simplex;

- suppose $f : X \to Y$ is a simplicial map between simplicial complexes X and Y and consider two points $x \in \mathrm{int}(\tau) \subset X$, $y \in \mathrm{int}(\sigma) \subset Y$ such that $f(x) = y$. Then $f(\tau) = \sigma$.

Let $z_1 \in \mathrm{int}(\sigma_1), z_2 \in \mathrm{int}(\sigma_2)$ be such that $z_1 \sim z_2$. It follows from Lemma 1.3 that there is a point $z \in Y_{\tau_1 \cap \tau_2}$ which glues together z_1 and z_2. The first fact above gives $z \in \mathrm{int}(\sigma)$ for some simplex $\sigma \subset Y_{\tau_1 \cap \tau_2}$. Since the embeddings $i_{\tau_1 \cap \tau_2, \tau_j}, j = 1, 2$ are simplicial maps, the second fact above gives $i_{\tau_1 \cap \tau_2, \tau_j}(\sigma) = \sigma_j$ for $j = 1, 2$. It follows from the last equality that $\pi(\sigma_1) = \pi(\sigma_2)$. $\square$

Comment. In the setting as in the statement of Lemma 1.6 we say that the simplex σ *glues together* the simplices σ_1 and σ_2.

Lemma 1.7. *Let A be an assembly system over a simplicial complex K, and let Y be the A-quotient. Then there is a natural structure of a simplicial complex on Y.*

Proof. We only give a sketch of the proof, and direct the reader's attention to the essential steps in giving the space Y a structure of a simplicial complex.
Since for every $\tau \subset K$ the space Y_τ is equipped with a distinguished family $S(Y_\tau)$ of geometric simplices, the space $\coprod_{\tau \subset K} Y_\tau$ is equipped with the family $\mathcal{Y} = \bigcup_{\tau \subset K} S(Y_\tau)$.

We want to realize geometric simplices in the A-quotient as subsets of the form $\pi(\sigma)$, where $\pi : \coprod_{\tau \subset K} Y_\tau \to Y$ is the quotient map and $\sigma \in \mathcal{Y}$. The characteristic maps of those simplices will be of the form $\pi \circ \phi_\sigma$, where $\phi_\sigma : \Delta^i \to \sigma$ is the characteristic map for σ which comes from the simplicial complex structure given on each of the spaces Y_τ. It follows from basic topology and Fact 1.4 that every such composition is a homeomorphism.

Note that there seems to be an ambiguity concerning the choice of a characteristic map for a simplex in Y whenever it is the image by π of more than one simplex. An easy argument shows though, that any two such maps differ by a precomposition through an affine isomorphism of the underlying standard simplex, and in our definition of a characteristic map we stated that we do not distinguish such maps.

Put $S(Y) = \{(\pi(\sigma), \pi \circ \phi_\sigma)\}_{\sigma \in \mathcal{Y}}$. We skip the details of checking that this family gives a structure of a simplicial complex on Y.

$\square$

Heading towards the end of this subsection, we will now describe a construction useful in the upcoming part of the paper, concerning continuous maps, whose domain is an $\mathcal{A}$-quotient Y.

Lemma 1.8. *Let $\mathcal{A}$ be an assembly system over some simplicial complex K, and let Y be the $\mathcal{A}$-quotient. Let Z be a topological space. Suppose that for every $\tau \in S(K)$ there is a continuous map $f_\tau : Y_\tau \to Z$ and the following condition holds: for $\rho \subset \tau \in S(K)$ we have $f_\rho = f_\tau \circ i_{\rho\tau}$. Then there is a unique continuous map $F : Y \to Z$ induced by the maps f_τ which makes the diagram below commute:*

$$\begin{tikzcd}
\coprod_{\tau\subset K} Y_\tau \arrow[d, "\pi"'] \arrow[dr, "f=\coprod f_\tau"] & \\
Y \arrow[r, dashed, "F"'] & Z
\end{tikzcd} \tag{3}$$

The proof is straightforward and we skip it.

1.2 Semi-barycentric maps

We now describe a type of mappings between simplicial complexes which is used in the description of Markov compacta in [Paw15], and we derive its basic properties.

Definition 1.9. Let X, Y be simplicial complexes. A map $f\colon X \to Y$ is called *semi-barycentric* if it satisfies the following conditions:

1. for every simplex $\sigma \subset X$ there is a simplex $\tau \subset Y$ such that $f(\sigma) \subset \tau$ and $f_{\restriction\sigma}$ is an affine map,
2. the vertices of X are mapped to the vertices of Y' (the first barycentric subdivision of Y),
3. for every simplex $\sigma \subset X$ there is a vertex $v \in Y$ such that $f(\sigma) \subset \mathrm{st}(v, Y')$, that is, the image of σ by f is contained in the closed star of v with respect to Y' as in Definition 0.4.

Remark 1.10. Let X, Y be simplicial complexes. Any simplicial map $f\colon X \to Y'$ is semi-barycentric as a map $X \to Y$.

Recall that a simplicial complex can be equipped with the standard piecewise linear metric, see [BH99], section $7A.10$. We refer to this metric when we speak of the diameter of a set in the lemma below.

Lemma 1.11. *Let $\mathcal{I} = (X_i, f_i)_{i\geq 0}$ be an inverse system, where for each $i \geq 0$:*

- *the space X_i is a simplicial complex;*
- *the map $f_i\colon X_{i+1} \to X_i$ is semi-barycentric.*

Moreover we assume that there is a global bound on the dimensions of the complexes X_i. Then $\mathcal{I}$ has mesh property in the sense of Definition 1.4 *in [Paw15], i.e. for any $i \geq 0$ we have*

$$\lim_{n\to\infty} \max_{F\in\mathcal{F}_n^i} \mathrm{diam} F = 0, \tag{4}$$

where

$$\mathcal{F}_n^i = \{(f_i \circ f_{i+1} \circ \ldots \circ f_{n-1} \circ f_n)(\sigma)\colon \sigma \textit{ is a simplex in } X_n\} \tag{5}$$

Proof. Let $i \geq 0$ and consider a simplex $\sigma \subset X_{i+1}$. From Condition 1 of Definition 1.9 we get a simplex $\tau \subset X_i$ such that $f_i(\sigma) \subset \tau$ and $f_{i\restriction\sigma}$ is an affine map. After precomposing with the characteristic maps ϕ_σ and ϕ_τ of the simplices σ and τ we may interpret $f_{i\restriction\sigma}$ as a map from $\Delta^{\dim\sigma}$ to the barycentric subdivision $(\Delta^{\dim\tau})'$ of $\Delta^{\dim\tau}$. Now, let the vertices $v_1, \ldots, v_l$ span σ and let the vertices $w_1, \ldots, w_m$ span τ. We can identify setwise the subsets of $\{w_1, \ldots, w_m\}$ with the faces of τ. For $j = 1, \ldots, l$ let $A_j \subset \{w_1, \ldots, w_m\}$ be the face whose barycenter coincides with $f_i(v_j)$. Then $f_{i\restriction\sigma}$ has the form

$$f((\lambda_1, \ldots, \lambda_l)) = \sum_{j=1}^{l} \frac{\lambda_j}{a_j} \sum_{i \in A_j} p_i, \tag{6}$$

where $a_j = |A_j|$ and $p_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the i-th position. Notice that Condition 3 of Definition 1.9 means exactly that $A_1 \cap \ldots \cap A_l \neq \emptyset$. It follows now from Lemma 1.13.1 in [Eng78] that

$$\mathrm{diam} f_{i\restriction\sigma}(\sigma) \leq \frac{\dim\tau}{\dim\tau + 1} \mathrm{diam}\sigma. \tag{7}$$

Thus we see that for a fixed i and n, for every simplex $\sigma \subset X_n$ we have

$$\mathrm{diam}(f_i \circ f_{i+1} \circ \ldots \circ f_{n-1} \circ f_n)(\sigma) \leq \left(\frac{k_{i,n}}{k_{i,n} + 1} \right)^{n-i} \mathrm{diam}\sigma \tag{8}$$

for some $k_{i,n}$, since we assumed a global bound on the dimensions of the complexes X_i. It is now clear that

$$\lim_{n\to\infty} \max_{F \in \mathcal{F}_n^i} \mathrm{diam} F = 0. \tag{9}$$

□

2 Simplicial assembly system determined by replacement rules and labelling

We now turn to the first step towards constructing Markov compacta. Given a simplicial complex K, we want to produce from it another (possibly more complicated) complex L, and do so in a controlled manner, so that there is a simplicial map from L to K. We will define a set of rules describing how to replace a simplex with a simplicial complex, and then "label" K with this set of rules. This will result with a simplicial assembly system over K, with an induced map from the A-quotient to K.

2.1 A good family of simplices

Definition 2.1. A *good family of simplices* is a pair $\mathcal{D} = \left[\Sigma, \{z_\beta\}_{\beta \in B}\right]$, where

- Σ is a set of simplices (not necessarily finite and not necessarily of pairwise distinct dimensions);

- B is the set of all proper faces in all simplices $\sigma \in \Sigma$;
- For every $\sigma \in \Sigma$ and every proper face $\beta \subset \sigma$, the map $z_\beta : \sigma_\beta \to \sigma$ is a simplicial embedding onto β for some $\sigma_\beta \in \Sigma$.

For a fixed $\beta \in B$ the map z_β is unique, and we assume also that the family $\{z_\beta\}_{\beta \in B}$ is closed under composition. Notice that this means that for proper faces $\beta \subset \sigma$ and $\alpha \subset \sigma_\beta$ the equality $z_\beta \circ z_\alpha = z_{z_\beta(\alpha)}$ holds.

Example 2.2. For a fixed $n \in \mathbb{N}$ define a family of simplices $\mathcal{D}_{\leq n}$ in the following way: let $\Sigma = \{\sigma_0, \sigma_1, \ldots, \sigma_n\}$ be a set consisting of one simplex per dimension from 0 up to n, where we assume that each simplex is equipped with an ordering of vertices. The family $\{z_\beta\}_{\beta \in B}$ is defined as follows: for each $k \leq n$ and a j-dimensional proper face $\beta \subset \sigma_k$ set $(\sigma_k)_\beta := \sigma_j$; then for $z_\beta : \sigma_\beta \to \sigma_k$ we choose the unique affine isomorphism $\sigma_j \to \beta \subset \sigma_k$ that respects the given orderings of vertices. It is clear that the family $\mathcal{D}_{\leq n}$ is a good family of simplices.

Example 2.3. Let K be an arbitrary simplicial complex. We can consider a family $\mathcal{D}_K$, in which $\Sigma = S(K)$, and for $\beta, \sigma \in S(K)$ such that $\beta \subset \sigma$ is a proper face of σ, we can set $\sigma_\beta = \beta$ and $z_\beta = id_\beta$. In this case it is also clear that this is a good family of simplices.

2.2 Rules of replacement

Let $\mathcal{D} = \left[\Sigma, \{z_\beta\}_{\beta \in B}\right]$ be a good family of simplices.

Definition 2.4. A *rule of replacement* for $\sigma \in \Sigma$ is a pair (P_σ, π_σ), where

- P_σ is a finite simplicial complex,
- $\sigma \xleftarrow{\pi_\sigma} P_\sigma$ is a semi-barycentric map.

A *good family of rules of replacement for* $\mathcal{D}$ is a pair $\mathcal{R}_\mathcal{D} = \left[\{(P_\sigma, \pi_\sigma) : \sigma \in \Sigma\}, \{P_{z_\beta} : \beta \in B\}\right]$, where $\{(P_\sigma, \pi_\sigma) : \sigma \in \Sigma\}$ is a family of rules of replacement such that every simplex from Σ is equipped with a single rule, and $\{P_{z_\beta} : \beta \in B\}$ is a family of *bonding maps* such that for $\sigma \in \Sigma$ and a proper face $\beta \subset \sigma$ (equipped with a map $z_\beta : \sigma_\beta \to \sigma$ for some $\sigma_\beta \in \Sigma$) the map $P_{z_\beta} : P_{\sigma_\beta} \to P_\sigma$ is a simplicial embedding. We assume that the family $\{P_{z_\beta}\}$ satisfies the following conditions:

1. For any $\sigma \in \Sigma$ and any proper face $\beta \subset \sigma$, the equality $\pi_\sigma \circ P_{z_\beta} = z_\beta \circ \pi_{\sigma_\beta}$ holds, in other words the diagram below is commutative:

$$\begin{tikzcd} \sigma_\beta \arrow[d, "z_\beta"'] & P_{\sigma_\beta} \arrow[l, "\pi_{\sigma_\beta}"'] \arrow[d, "P_{z_\beta}"] \\ \sigma & P_\sigma \arrow[l, "\pi_\sigma"] \end{tikzcd} \tag{10}$$

We also demand that the equality $\pi_\sigma^{-1}(\beta) = P_{z_\beta}(P_{\sigma_\beta})$ holds.

2. For proper faces $\beta \subset \sigma$ and $\alpha \subset \sigma_\beta$, the equality $P_{z_\beta} \circ P_{z_\alpha} = P_{z_{z_\beta(\alpha)}}$ holds.

If possible, and if it will not lead to ambiguities, we will skip the index and write just $\mathcal{R}$ instead of $\mathcal{R}_\mathcal{D}$. Of course, a fixed good family $\mathcal{D}$ may admit more than one good family of rules of replacement.

Example 2.5. In this example, and further in Examples 2.8, 2.12, 3.5 and 3.8 we will give a formal and precise description of the idea shown in Example 3.5 in [BN17]. Consider a good family $\mathcal{D}$, in which Σ consists of one 0-simplex σ_0 and two 1-simplices σ_1^1 and σ_1^2. Denote the vertices of σ_1^1 by s and t, and vertices of σ_1^2 by x and y. For all $\beta \in B$ the map z_β takes σ_0 to β.

Let us define a good family $\mathcal{R}$ of rules of replacement for the good family $\mathcal{D}$, which consists of the following rules:

- As P_{σ_0} take a space consisting of two 0-simplices a and b, and set π_{σ_0} as the map that takes a and b to σ_0;
- The space $P_{\sigma_1^1}$ is the disjoint union of two barycentrically subdivided 1-simplices τ and ρ, and the map $\pi_{\sigma_1^1}$ is a simplicial map induced by taking the vertices γ and δ of τ to the vertices s and t respectively, while the vertices η and θ of ρ are taken to s and t respectively.
- the space $P_{\sigma_1^2}$ and the map $\pi_{\sigma_1^2}$ are defined similarly, this time with the 1-simplices denoted by μ and ν, and their vertices by ϕ, ψ and ζ, ξ respectively.

Let us now describe bonding maps for $\mathcal{R}$.

- For the map z_s set P_{z_s} as the embedding which takes a to γ and b to η;
- for the map z_t set P_{z_t} as the embedding which takes a to δ and b to θ;
- for the map z_x set P_{z_x} as the embedding which takes a to ϕ and b to ζ;
- for the map z_y set P_{z_y} as the embedding which takes a to ξ and b to ψ.

Example 2.6. Let K, L be simplicial complexes and let $h : L \to K$ be a semi-barycentric map. Let us define a good family of rules $\mathcal{R}_h$ for the good family $\mathcal{D}_K$ from Example 2.3: for a simplex $\sigma \in \Sigma = S(K)$ set $P_\sigma := h^{-1}(\sigma)$, and set $\pi_\sigma := h_{\restriction h^{-1}(\sigma)}$. Of course for $\beta \subset \sigma$ we set $z_\beta = id_\beta$ and $P_{z_\beta} = id_{h^{-1}(\beta)} : h^{-1}(\beta) \to h^{-1}(\sigma)$. Obviously, P_σ is a subcomplex of L, and $\pi_\sigma \colon P_\sigma \to \sigma$ is semi-barycentric.

2.3 Labelling

We now proceed to describe how to build an assembly system over a simplicial complex X. Intuitively, we want to "label" simplices of X with a good family $\mathcal{D}$ and use corresponding rules of replacement from a good family $\mathcal{R}_\mathcal{D}$, so as to create an assembly system over X with the spaces Y_τ being the replacing complexes from the family $\mathcal{R}_\mathcal{D}$.

Definition 2.7. Let $\mathcal{D}$ be a good family of simplices. A $\mathcal{D}$-*labelling* of a simplicial complex X is a pair $\Lambda = \big[\lambda, \{u_\sigma\}_{\sigma \in S(X)}\big]$ where

- $\lambda : S(X) \to \Sigma$ is a map between the sets of simplices such that for every $\sigma \in S(X)$ the equality $\dim(\sigma) = \dim(\lambda(\sigma))$ holds,

- for every $\sigma \in S(X)$ the map $u_\sigma : \sigma \to \lambda(\sigma)$ is an isomorphism of simplices (identification of σ with its label $\lambda(\sigma)$), where we assume the following condition:
$$\text{for} \quad \rho \subset \tau \quad \text{we have} \quad \lambda(\rho) = \lambda(\tau)_{u_\tau(\rho)} \tag{11}$$
(the symbol $\lambda(\tau)_{u_\tau(\rho)}$ is to be understood as the symbol σ_β from Definition 2.1: $u_\tau(\rho)$ is a proper face of $\lambda(\tau)$ corresponding to the face ρ of τ). Moreover we demand the diagram below to be commutative:
$$\begin{tikzcd} \rho \arrow[r, hook] \arrow[d, "u_\rho"'] & \tau \arrow[d, "u_\tau"] \\ \lambda(\rho) \arrow[r, "z_{u_\tau(\rho)}"'] & \lambda(\tau) \end{tikzcd} \tag{12}$$

Example 2.8. Let X be a triangulation of the circle S^1 with three 1-simplices e_1, e_2, e_3. Denote the vertices of this cycle by a, b and c so that $e_1 = [a, b]$, $e_2 = [b, c]$ and $e_3 = [c, a]$.

Consider a $\mathcal{D}$-labelling of X with the good family $\mathcal{D}$ described in Example 2.5, of the following form: for the 0-simplices a, b, c set $\lambda(a) = \lambda(b) = \lambda(c) = \sigma_0$, and for the 1-simplices set $\lambda(e_1) = \lambda(e_2) = \sigma_1^1$, $\lambda(e_3) = \sigma_1^2$. For every $\sigma \in S(X)$ set $u_\sigma : \sigma \to \lambda(\sigma)$ as any simplicial isomorphism between σ and $\lambda(\sigma)$. It is clear that conditions (11) and (12) are satisfied.

Example 2.9. (Tautological labelling) Let K be a simplicial complex and consider the good family $\mathcal{D}_K$ as described in Example 2.3. Consider the pair $\Lambda = \left[id_{S(K)}, \{id_\sigma\}_{\sigma \in S(K)}\right]$. Of course this is a $\mathcal{D}_K$-labelling of K.

Example 2.10. (Pullback labelling) Let K, L be simplicial complexes, Λ be a $\mathcal{D}$-labelling of the complex K for some good family of simplices $\mathcal{D}$ and let $h : L \to K$ be a non-degenerate simplicial map. We can consider the following $\mathcal{D}$-labelling Λ^p of the complex L: for $\sigma \in S(L)$ set $\lambda^p(\sigma) := \lambda(h(\sigma))$ and $u^p_\sigma := u_{h_{\restriction\sigma}(\sigma)} \circ h_{\restriction\sigma}$. It is clear that this is a labelling in the sense of Definition 2.7.

2.4 Construction

In this section we show how to carry out the first step in the construction of an inverse sequence associated with a Markov compactum. We form a simplicial assembly system A over some complex X_0, an A-quotient Y and a semi-barycentric map $Y \to X_0$, out of the following data: a good family $\mathcal{D}$, a family of rules of replacement $\mathcal{R}$ for $\mathcal{D}$ and a $\mathcal{D}$-labelling Λ of X_0. This is an essential part of the construction as having carried out one step we may hope to iterate it to create an inverse system of simplicial complexes and maps between them in a unique way.

Let $\mathcal{D} = \left[\Sigma, \{z_\beta\}_{\beta \in B}\right]$ be a good family of simplices, X_0 a simplicial complex, $\mathcal{R} = \left[\{(P_\sigma, \pi_\sigma) : \sigma \in \Sigma\}, \{P_{z_\beta} : \beta \in B\}\right]$ be a good family of rules of replacement for $\mathcal{D}$ and let $\Lambda = \left[\lambda, \{u_\sigma\}_{\sigma \in S(X_0)}\right]$ be a $\mathcal{D}$-labelling of X_0. We set a structure of an assembly system $A = A(X_0, \mathcal{D}, \mathcal{R}, \Lambda)$ over X_0 in the following way.

For each simplex $\tau \subset X_0$ set

$$Y_\tau := P_{\lambda(\tau)} \tag{13}$$

where $\lambda(\tau)$ is the label of τ with respect to the labelling Λ (for each $\tau \subset X_0$ there is a seperate copy of the complex $P_{\lambda(\tau)}$).
We now proceed to define the embeddings $i_{\rho\tau}$. Let $\rho, \tau \subset X_0$ be such that $\rho \subset \tau$, and let $\lambda(\rho), \lambda(\tau)$ be their labels with respect to the labelling Λ. It follows from (10) and (12) that the diagram below is commutative:

$$\begin{tikzcd} \rho \arrow[r, "u_\rho"'] \arrow[d, hook] & \lambda(\rho) \arrow[d, dashed, "z_{u_\tau(\rho)}"] & P_{\lambda(\rho)} \arrow[l] \arrow[d, "P_{z_{u_\tau(\rho)}}"] \\ \tau \arrow[r, "u_\tau"] & \lambda(\tau) & P_{\lambda(\tau)} \arrow[l] \end{tikzcd} \tag{14}$$

where $P_{z_{u_\tau(\rho)}}$ is the bonding map from the family $\mathcal{R}$. For each pair of simplices $\rho \subset \tau$ in X_0 define $i_{\rho\tau} := P_{z_{u_\tau(\rho)}}$.

Let us check that so defined maps $i_{\rho\tau}$ satisfy the composition property. Let $\rho \subset \tau \subset \nu$ be simplices in X_0 such that $\rho \subset \tau \subset \nu$. It now follows from (14) that the diagram below is commutative:

$$\begin{tikzcd} \rho \arrow[r, hook] \arrow[d, "u_\rho"'] & \tau \arrow[r, hook] \arrow[d, "u_\tau"] & \nu \arrow[d, "u_\nu"] \\ \lambda(\rho) \arrow[r, "z_{u_\tau(\rho)}"'] & \lambda(\tau) \arrow[r, "z_{u_\nu(\tau)}"'] & \lambda(\nu) \\ P_{\lambda(\rho)} \arrow[u, "\pi_{\lambda(\rho)}"] \arrow[r, "P_{z_{u_\tau(\rho)}}"'] & P_{\lambda(\tau)} \arrow[u, "\pi_{\lambda(\tau)}"'] \arrow[r, "P_{z_{u_\nu(\tau)}}"'] & P_{\lambda(\nu)} \arrow[u, "\pi_{\lambda(\nu)}"'] \end{tikzcd} \tag{15}$$

Now condition *2* imposed on the family $\{P_{z_\beta}\}$ in Definition 2.4 leads to the following computation:

$$i_{\rho\nu} = P_{z_{u_\nu(\rho)}} = P_{z_{z_{u_\nu(\tau)}(u_\tau(\rho))}} = P_{z_{u_\tau(\rho)}} \circ P_{z_{u_\nu(\tau)}} = i_{\tau\nu} \circ i_{\rho\tau}. \tag{16}$$

which proves that maps $i_{\rho\tau}$ satisfy the composition property. Let us now check that the intersection property is satisfied. Let $\tau_1, \tau_2, \sigma \subset X_0$ be such that $\tau_1, \tau_2 \subset \sigma$. We want to show that $i_{\tau_1\sigma}(Y_{\tau_1}) \cap i_{\tau_2\sigma}(Y_{\tau_2}) = i_{\tau_1\cap\tau_2\sigma}(Y_{\tau_1\cap\tau_2})$. It follows from condition 1 stated in Definition 2.4 that

$$\begin{aligned} i_{\tau_1\sigma}(Y_{\tau_1}) \cap i_{\tau_2\sigma}(Y_{\tau_2}) = P_{z_{u_\sigma(\tau_1)}}(P_{\lambda(\tau_1)}) \cap P_{z_{u_\sigma(\tau_2)}}(P_{\lambda(\tau_2)}) = \pi^{-1}_{\lambda(\sigma)}(\lambda(\tau_1)) \cap \pi^{-1}_{\lambda(\sigma)}(\lambda(\tau_2)) = \\ = \pi^{-1}_{\lambda(\sigma)}(\lambda(\tau_1) \cap \lambda(\tau_2)) = P_{z_{u_\sigma(\tau_1\cap\tau_2)}}(P_{\lambda(\tau_1\cap\tau_2)}) = i_{\tau_1\cap\tau_2\sigma}(Y_{\tau_1\cap\tau_2}). \end{aligned} \tag{17}$$

This way we have shown that we can truly speak of an assembly system

$$A = \left(\{Y_\tau\}_{\tau\in S(X_0)}, \{i_{\rho\tau}\}_{\rho\subset\tau}\right) \tag{18}$$

over X_0, as described above.
Put

$$X_1 := \coprod_{\tau\subset X_0} Y_\tau \big/ \sim \tag{19}$$

to be the A-quotient for the above described assembly system $A(X_0, \mathcal{D}, \mathcal{R}, \Lambda)$. It follows from Lemma 1.7 that X_1 is a simplicial complex.

Accordingly to the framework presented in Lemma 1.8 we will now describe a semi-barycentric map whose domain is the A-quotient X_1, and whose image is X_0. For every $\tau \in S(X_0)$ set $f_\tau : Y_\tau \to X_0$ as $f_\tau := u_\tau^{-1} \circ \pi_{\lambda(\tau)}$. Let us check that the maps f_τ satisfy the compatibility condition stated in Lemma 1.8. Let $\rho \in S(X_0)$ be a simplex such that $\rho \subset \tau$. We have

$$f_\tau \circ i_{\rho\tau} = u_\tau^{-1} \circ \pi_{\lambda(\tau)} \circ P_{z_{u_\tau(\rho)}} \overset{(14)}{=} u_\tau^{-1} \circ z_{u_\tau(\rho)} \circ \pi_{\lambda(\rho)} \overset{(14)}{=} u_\rho^{-1} \circ \pi_{\lambda(\rho)} = f_\rho. \tag{20}$$

Since A is a simplicial assembly system, X_0 is a simplicial complex and each of the maps f_τ is a semi-barycentric map, an easy argument shows that the map $\Pi_0 : X_1 \to X_0$ (induced by the maps f_τ) is then semi-barycentric.

Definition 2.11. Given a simplicial complex X_0 equipped with a $\mathcal{D}$-labelling Λ and a good family $\mathcal{R}$ of rules of replacement for $\mathcal{D}$, the simplicial complex X_1 constructed above is called the $(\Lambda, \mathcal{D}, \mathcal{R})$*-quotient over* X_0, and the map Π_0 is called the $(\Lambda, \mathcal{D}, \mathcal{R})$*-map from* X_1 *to* X_0.

Example 2.12. Consider a simplicial complex X equipped with a $\mathcal{D}$-labelling as in Example 2.8. Carry out the construction described above in this section. As a result the A-quotient Y is a barycentrically subdivided cycle of length 6. Notice that had we labelled all the 1-simplices of X with the simplex σ_1^1, the resulting A-quotient Y_2 would be the union of two disjoint and barycentrically subdivided copies of X. The reason for Y being connected lies in the "twist" carried out by the map P_{z_y} (as compared with the map P_{z_x}).

Example 2.13. Let K, L be simplicial complexes and let $h : L \to K$ be a semi-barycentric map. Consider a tautological $\mathcal{D}_K$-labelling Λ of the complex K as in Example 2.9 and a family $\mathcal{R}_h$ of rules of replacement for $\mathcal{D}_K$ as in Example 2.6. Then the $(\Lambda, \mathcal{D}_K, \mathcal{R}_h)$-quotient over K for the assembly system $A(K, \mathcal{D}_K, \mathcal{R}_h, \Lambda)$ can be identified with the complex L and the $(\Lambda, \mathcal{D}, \mathcal{R})$-map Π_0 coincides with h.

3 Constructive Markov compacta

Let $\mathcal{D}_a = \left[\Sigma_a, \{z^a_\beta\}_{\beta\in B_a}\right], \mathcal{D}_b = \left[\Sigma_b, \{z^b_\beta\}_{\beta\in B_b}\right]$ be good families of simplices and let $\mathcal{R}$ be a good family of rules of replacement for $\mathcal{D}_a$.

Definition 3.1. A $\mathcal{D}_b$*-labelling of the family* $\mathcal{R}$ is a $\mathcal{D}_b$-labelling

$$\Lambda_{\mathcal{R}} = \left[\lambda_{\mathcal{R}}, \{u_\tau^{\mathcal{R}}\}_{\tau \in S(\coprod_{\sigma \subset \Sigma_a} P_\sigma)}\right] \tag{21}$$

of the complex $\coprod_{\sigma \subset \Sigma_a} P_\sigma$ such that for simplices $\sigma_\beta, \sigma \in \Sigma_a$ such that $z_\beta^a : \sigma_\beta \to \sigma$ is the embedding from the good family $\mathcal{D}_a$ and for any two simplices $\rho_1 \subset P_{\sigma_\beta}, \rho_2 \subset P_\sigma$ such that $P_{z_\beta}(\rho_1) = \rho_2$ we have $\lambda_{\mathcal{R}}(\rho_1) = \lambda_{\mathcal{R}}(\rho_2)$ and the diagram

$$\begin{tikzcd} \rho_1 \arrow[rr, "P_{z_\beta}"] \arrow[dr, "u_{\rho_1}^{\mathcal{R}}"'] & & \rho_2 \arrow[dl, "u_{\rho_2}^{\mathcal{R}}"] \\ & \lambda_{\mathcal{R}}(\rho_1) = \lambda_{\mathcal{R}}(\rho_2) & \end{tikzcd} \tag{22}$$

commutes.

Remark 3.2. When $\mathcal{D}_a = \mathcal{D}_b = \mathcal{D}$ we will simply say that $\Lambda_{\mathcal{R}}$ is a $\mathcal{D}$-labelling of the family $\mathcal{R}$.

Lemma 3.3. *Let X_0 be a simplicial complex equipped with a $\mathcal{D}$-labelling Λ and let $\mathcal{R}$ be a good family of rules of replacement for $\mathcal{D}$. Then a $\mathcal{D}$-labelling $\Lambda_{\mathcal{R}}$ of the family $\mathcal{R}$ induces a $\mathcal{D}$-labelling of the $(\Lambda, \mathcal{D}, \mathcal{R})$-quotient.*

Proof. Let X_1 be the $(\Lambda, \mathcal{D}, \mathcal{R})$-quotient, and let $\pi : \coprod_{\tau \subset X_0} Y_\tau \to X_1$ be the quotient map. We will show that on X_1 there exists a unique $\mathcal{D}$-labelling $\Lambda_1 = \left[\lambda_1, \{u_\sigma^1\}_{\sigma \in S(X_1)}\right]$ such that if for any simplices $\sigma_1 \subset \coprod_{\tau \subset X_0} Y_\tau$ and $\sigma \subset Y$ we have $\pi(\sigma_1) = \sigma$, then $\lambda_1(\sigma) = \lambda_{\mathcal{R}}(\sigma_1)$ and $u_\sigma^1 \circ \pi = u_{\sigma_1}^{\mathcal{R}}$.

By the above conditions the uniqueness of such a map is clear, but the well-definedness of Λ_1 needs checking.

First notice that a $\mathcal{D}$-labelling $\Lambda_{\mathcal{R}}$ of the family $\mathcal{R}$ induces in a natural way a $\mathcal{D}$-labelling of the complex $\coprod_{\tau \subset X_0} Y_\tau$, which we will also denote by $\Lambda_{\mathcal{R}}$. Let us begin by checking that if for simplices $\sigma_1, \sigma_2 \subset \coprod_{\tau \subset X_0} Y_\tau$ we have $\pi(\sigma_1) = \pi(\sigma_2) = \sigma$, then $\lambda_{\mathcal{R}}(\sigma_1) = \lambda_{\mathcal{R}}(\sigma_2)$. Let $\sigma_1 \subset Y_\rho, \sigma_2 \subset Y_\tau$ for some $\rho, \tau \subset X_0$, $\sigma_1 = i_{\rho \cap \tau\, \rho}(\varsigma), \sigma_2 = i_{\rho \cap \tau\, \tau}(\varsigma)$ where $\varsigma \subset Y_{\rho \cap \tau}$ glues together σ_1 and σ_2. By (22) we have

$$\lambda_{\mathcal{R}}(\sigma_1) = \lambda_{\mathcal{R}}(i_{\rho \cap \tau\, \rho}(\varsigma)) = \lambda_{\mathcal{R}}(\varsigma) = \lambda_{\mathcal{R}}(i_{\rho \cap \tau\, \tau}(\varsigma)) = \lambda_{\mathcal{R}}(\sigma_2). \tag{23}$$

Thus there exists a unique map $\lambda_1 : S(X_1) \to \Sigma$ such that $\lambda_1 \circ \pi = \lambda_{\mathcal{R}}$.
Now let us check that the maps $\{u_\sigma^1\}_{\sigma \in S(X_1)}$ are well-defined. For simplices σ_1, σ_2 as above this means that the equality $u_{\sigma_1}^{\mathcal{R}} = u_{\sigma_2}^{\mathcal{R}}$ should hold. Indeed, by (22) we get

$$u_{\sigma_1}^{\mathcal{R}} = u_{\sigma_1}^{\mathcal{R}} \circ i_{\tau_1 \cap \tau_2 \tau_1} = u_\varsigma^{\mathcal{R}} = u_{\sigma_1}^{\mathcal{R}} \circ i_{\tau_1 \cap \tau_2 \tau_2} = u_{\sigma_2}^{\mathcal{R}} \tag{24}$$

It remains to check conditions (11) and (12) stated in Definition 2.7, but they follow easily from (22), (23) and (24) above, and we skip the details. □

A more general version of the above Lemma is used later in this paper, and we formulate it as a remark.

Remark 3.4. Let $\mathcal{D}_a, \mathcal{D}_b$ be good families of simplices. Let X_0 be a simplicial complex equipped with a $\mathcal{D}_a$-labelling Λ and let $\mathcal{R}$ be a good family of rules of replacement for $\mathcal{D}_a$. Then a $\mathcal{D}_b$-labelling of the family $\mathcal{R}$ induces a $\mathcal{D}_b$-labelling of the $(\Lambda, \mathcal{D}_a, \mathcal{R})$-quotient.

Proof. The proof is almost identical to the proof of Lemma 3.3 and we skip it. □

Example 3.5. Consider the good family $\mathcal{D}$ and the family $\mathcal{R}$ of rules of replacement for $\mathcal{D}$ from Example 2.5 and the complex X equipped with the $\mathcal{D}$-labelling from Example 2.8. Define a $\mathcal{D}$-labelling $\Lambda_{\mathcal{R}}$ of $\coprod_{\sigma \subset \Sigma} P_\sigma$ in the following way: for any seven of the 1-simplices $\tau \in S(\coprod_{\sigma \subset \Sigma} P_\sigma)$ set $\lambda_{\mathcal{R}}(\tau) = \sigma_1^1$, and for the only remaining 1-simplex ρ set $\lambda_{\mathcal{R}}(\rho) = \sigma_1^2$; further for all 0-simplices σ set $\lambda(\sigma) = \sigma_0$. Then for every $\sigma \in S(\coprod_{\sigma \subset \Sigma} P_\sigma)$ set $u_\sigma : \sigma \to \lambda(\sigma)$ as any simplicial isomorphism between σ and $\lambda(\sigma)$. Let $A = A(X, \mathcal{D}, \mathcal{R}, \Lambda)$ be the simplicial assembly system, let X_1 be the $(\Lambda, \mathcal{D}, \mathcal{R})$-quotient and let $\pi : \coprod_{\tau \subset X} Y_\tau \to X_1$ be the quotient map as described in Section 2.1. It follows from Lemma 3.3 that $\Lambda_{\mathcal{R}}$ induces a $\mathcal{D}$-labelling Λ_1 of X_1 such that for the distinguished 1-simplex ρ above, its image by the map π is labelled with σ_1^2, images of the remaining 1-simplices are labelled with σ_1^1, and all 0-simplices are labelled with σ_0, with the appropriate family of isomorphisms $\{u_\sigma^1\}$ identifying them with elements of the good family $\mathcal{D}$ of the form described in the Lemma

Definition 3.6. Let $\mathcal{D}$ be a finite good family of simplices, $\mathcal{R}$ be a family of rules of replacement for $\mathcal{D}$, X_0 be a finite simplicial complex equipped with a $\mathcal{D}$-labelling Λ, and finally let $\Lambda_{\mathcal{R}}$ be a $\mathcal{D}$-labelling of the family $\mathcal{R}$. Consider the inverse system

$$\mathcal{I} = \mathcal{I}(X_0, \mathcal{D}, \mathcal{R}, \Lambda, \Lambda_{\mathcal{R}}) = \big(\{X_i : i \geq 0\}, \{\Pi_i : i \geq 0\}\big), \tag{25}$$

in which the complexes X_i (equipped with auxiliary $\mathcal{D}$-labellings Λ_i) and maps Π_i are defined recursively in the following manner:

- for $i = 0$ set $X_i := X_0$, and set $\Lambda_i := \Lambda$;
- assuming we have already defined the complex X_i equipped with its $\mathcal{D}$-labelling Λ_i, define X_{i+1} as the $(\Lambda_i, \mathcal{D}, \mathcal{R})$-quotient over X_i, along with the $\mathcal{D}$-labelling Λ_{i+1} induced by the $\mathcal{D}$-labelling of the family $\mathcal{R}$ as shown in Lemma 3.3;
- for each $i \geq 0$ we also obtain a $(\Lambda_i, \mathcal{D}, \mathcal{R})$-map $\Pi_i : X_{i+1} \to X_i$ (which is semi-barycentric).

A *constructive Markov system* is any inverse system $\mathcal{I}(X_0, \mathcal{D}, \mathcal{R}, \Lambda, \Lambda_{\mathcal{R}})$ of the form described above.

Definition 3.7. A *constructive Markov compactum* is the inverse limit of any constructive Markov system.

Example 3.8. Consider the good family $\mathcal{D}$ and the family $\mathcal{R}$ of rules of replacement for $\mathcal{D}$ as in Example 2.5, the complex X equipped with the $\mathcal{D}$-labelling as in Example 2.8 and the $\mathcal{D}$-labelling $\Lambda_{\mathcal{R}}$ of $\mathcal{R}$ as in Example 3.5. It can be shown that for each $i \geq 0$ the space X_i is a circle subdivided into $3 \cdot 4^i$ 1-simplices, and the map $\Pi_i \colon X_{i+1} \to X_i'$ is topologically a 2-fold covering. It follows that the constructive Markov compactum obtained as the limit of the Markov system $\mathcal{I}(X, \mathcal{D}, \mathcal{R}, \Lambda, \Lambda_{\mathcal{R}})$ is a solenoid.

4 Comparison with earlier definitions of Markov compacta

In this section we study the relationship of the concept of constructive Markov compactum, as described above in this paper, to some previously described in the literature concept of Markov compacta.

4.1 Earlier definitions describe constructive Markov compacta

In this subsection we discuss the earlier definitions of Markov compacta and make an important observation about their relationship to constructive Markov compacta as defined in this paper, see Proposition 4.10.

Definition 4.1. [Paw15] A *Markov system* is an inverse system $(K_i, f_i)_{i\geq 0}$, where

- for every $i \geq 0$, the space K_i is a finite simplicial complex, and we have $\sup_i \dim K_i < \infty$;
- for every simplex $\sigma \in K_{i+1}$ its image $f_i(\sigma)$ is contained in some simplex belonging to K_i and the restriction $f_{i\restriction\sigma}$ is an affine map;
- simplices in $\coprod K_i$ can be assigned finitely many *types* so that for any simplices $\sigma \subset K_i$ and $\tau \subset K_j$ of the same type there exist type-preserving isomorphisms of subcomplexes $i_k^{\sigma,\tau} : (f_i^{i+k})^{-1}(\sigma) \to (f_j^{j+k})^{-1}(\tau)$ for $k \geq 0$ such that the following diagram commutes:

$$\begin{tikzcd}
\sigma \arrow[d, "i_0^{\sigma,\tau}"] & f_i^{-1}(\sigma) \arrow[l, "f_i"'] \arrow[d, "i_1^{\sigma,\tau}"] & \dots \arrow[l] & (f_i^{i+k})^{-1}(\sigma) \arrow[l] \arrow[d, "i_k^{\sigma,\tau}"] & (f_i^{i+k+1})^{-1}(\sigma) \arrow[l, "f_{i+k}"'] \arrow[d, "i_{k+1}^{\sigma,\tau}"] & \dots \arrow[l] \\
\tau & f_j^{-1}(\sigma) \arrow[l, "f_j"'] & \dots \arrow[l] & (f_j^{j+k})^{-1}(\tau) \arrow[l] & (f_j^{j+k+1})^{-1}(\tau) \arrow[l, "f_{j+k}"'] & \dots \arrow[l]
\end{tikzcd} \tag{26}$$

where by f_b^a (for $a > b$) we denote the composition $f_b \circ f_{b+1} \circ \cdots \circ f_{a-1} : K_a \to K_b$.

Definition 4.2. [Paw15] A topological space $\mathcal{M}$ is a *Markov compactum* if it is the limit of a Markov system.

Definition 4.3. [Paw15] A Markov system (K_i, f_i) is called *barycentric* if, for any $i \geq 0$, the vertices of K_{i+1} are mapped by f_i to the vertices of the first barycentric subdivision K_i' of K_i.

Definition 4.4. [Paw15] A Markov system (K_i, f_i) has the *distinct types property* if for any $i \geq 0$ and any simplex $\sigma \in K_i$ all simplices in the pre-image $f_i^{-1}(\sigma)$ have pairwise distinct types.

We summarize the series of the above definitions in the following.

Definition 4.5. A topological space $\mathcal{M}$ is a *finitistic* Markov compactum if it is the limit of a Markov system which is barycentric and satisfies distinct types property.

The justification for the term "finitistic" comes from Remark 1.8 in [Paw15], that the associated Markov system is then determined by some finite initial data. The significance of this concept of a Markov compactum comes from the following result, Theorem 0.1, the main theorem of [Paw15].

Theorem 4.6. *[Paw15] Let G be a hyperbolic group. Then, the Gromov boundary ∂G of G is homeomorphic to a Markov compactum $\varprojlim K_i$ defined by a Markov system $(K_i, f_i)_{i\geq 0}$. Moreover, we can require (simultaneously) that:*

- *the system $(K_i, f_i)_{i\geq 0}$ is barycentric and satisfies distinct types property and mesh property;*
- *the dimensions of the complexes K_i are bounded from above by the topological dimension $dim\partial G$.*

Remark 4.7. The maps in the Markov system $(K_i, f_i)_{i\geq 0}$ appearing in the Pawlik's proof of Theorem 4.6 in [Paw15] are actually semi-barycentric in the sense of Definition 1.9. This can be checked upon inspecting Definition 4.5 and Remark 4.6 in [Paw15], where the maps f_i are defined. One can see that conditions labeled *(ii)* and *(iii)* in Definition 4.5 there coincide with Conditions 1 and 2 from Definition 1.9. Remark 4.6 in [Paw15] discusses what we call Condition 3.

In light of the above remark, we reformulate Definitions 4.1 - 4.5 and Theorem 4.6 to make a record of its slightly stronger version (Theorem 4.8) actually proved in [Paw15].

Definition 4.8. A *semi-barycentric Markov system* is an inverse system $(K_i, f_i)_{i\geq 0}$, where

- for every $i \geq 0$, the space K_i is a finite simplicial complex, and we have $\sup_i \dim K_i < \infty$;
- for every $i \geq 0$ the map $f_i\colon X_{i+1} \to X_i$ is semi-barycentric;
- simplices in $\coprod K_i$ can be assigned finitely many *types* so that for any simplices $\sigma \subset K_i$ and $\tau \subset K_j$ of the same type there exist type-preserving isomorphisms of subcomplexes $i_k^{\sigma,\tau} : (f_i^{i+k})^{-1}(\sigma) \to (f_j^{j+k})^{-1}(\tau)$ for $k \geq 0$ such that the following diagram commutes:

$$\begin{tikzcd}
\sigma \arrow[d, "i_0^{\sigma,\tau}"] & f_i^{-1}(\sigma) \arrow[l, "f_i"'] \arrow[d, "i_1^{\sigma,\tau}"] & \dots \arrow[l] & (f_i^{i+k})^{-1}(\sigma) \arrow[l] \arrow[d, "i_k^{\sigma,\tau}"] & (f_i^{i+k+1})^{-1}(\sigma) \arrow[l, "f_{i+k}"'] \arrow[d, "i_{k+1}^{\sigma,\tau}"] & \dots \arrow[l] \\
\tau & f_j^{-1}(\sigma) \arrow[l, "f_j"'] & \dots \arrow[l] & (f_j^{j+k})^{-1}(\tau) \arrow[l] & (f_j^{j+k+1})^{-1}(\tau) \arrow[l, "f_{j+k}"'] & \dots \arrow[l]
\end{tikzcd} \tag{27}$$

where by f_b^a (for $a > b$) we denote the composition $f_b \circ f_{b+1} \circ \dots \circ f_{a-1} : K_a \to K_b$.

A semi-barycentric Markov system as above satisfies the *distinct types property* if for any $i \geq 0$ and any simplex $\sigma \in K_i$ all simplices in the pre-image $f_i^{-1}(\sigma)$ have pairwise distinct types.

Theorem 4.6 can be now reformulated in the following, more precise form (and it holds by the same argument given in [Paw15] due to the comment in Remark 4.7).

Theorem 4.9. *Let G be a hyperbolic group. Then, the Gromov boundary ∂G of G is homeomorphic to the limit of a semi-barycentric Markov system with the distinct types property.*

Our aim in this subsection is to prove the following.

Proposition 4.10. *For any semi-barycentric Markov system $(K_i, f_i)_{i\geq 0}$ with the distinct types property, its subsequence $(K_i, f_i)_{i\geq 1}$ obtained by omitting the first term K_0 is isomorphic to some constructive Markov system.*

In view of Theorem 4.9 the above proposition yields the following.

Corollary 4.11. The Gromov boundary ∂G of any hyperbolic group G is homeomorphic to some constructive Markov compactum.

Proof. (of Proposition 4.10)

We describe a constructive Markov system $\mathcal{I} = \mathcal{I}(X_0, \mathcal{D}, \mathcal{R}, \Lambda, \Lambda_{\mathcal{R}})$ as in the assertion in the following way.

1. Set $\mathcal{D} = (\Sigma, \{z_\beta\})$, where Σ consists of one representative of each type of simplices in $\coprod K_i, i \geq 1$, and the family $\{z_\beta\}$ is of the following form. Let $\beta \subset \sigma \in \Sigma$ be a proper face; set σ_β as the representative of the type of β. Since σ_β and β have the same type, there is an isomorphism $i_0 : \sigma_\beta \to \beta$ as in the last part of Definition 4.1 (where the notation $i_0^{\sigma,\tau}$ is used), and it is unique, because it is type preserving, and the types of faces of both σ_β and β are pairwise distinct. It's here that we use the fact that σ_β and β are simplices in $\coprod K_i$, where $i \geq 1$, and that the distinct types property holds in $(K_i, f_i)_{i\geq 0}$. Put $z_\beta = i_0$ and view it as an embedding of σ_β into σ. Define the maps z_β analogously for all proper faces of all simplices in Σ. It follows easily that for a fixed β the map z_β is unique. Moreover, from the distinct types property it follows that the family $\{z_\beta\}_{\beta\in B}$ is closed under composition.

2. Set $X_0 := K_1$.

3. We proceed to define a $\mathcal{D}$-labelling Λ on X_0. For every simplex $\tau \in S(X_0)$ set $\lambda(\tau)$ to be the representative of the type of τ in the family $\mathcal{D}$. Further set u_τ to be the unique isomorphism $i_0 : \tau \to \lambda(\tau)$ as in the last part of Definition 4.1. Using the uniqueness of the maps u_τ and the fact that Σ consist of exactly one representative of each type of simplices, it is fairly easy to check that this definition of Λ satisfies the conditions stated in Definition 2.7, and we skip the details.

4. The family $\mathcal{R}$ for $\mathcal{D}$ has the following form. Consider a simplex $\sigma \in \Sigma$. Then $\sigma \subset K_n$ for some n. Define $P_\sigma := f_n^{-1}(\sigma)$, and $\pi_\sigma := f_{n\restriction f_n^{-1}(\sigma)}$. Define the rules of replacement analogously for all $\sigma \in \Sigma$. It remains to describe bonding maps for such rules. Let $\sigma_\beta, \sigma \in \Sigma$ be such that there is an embedding $z_\beta : \sigma_\beta \to \sigma$. Then $\sigma_\beta \subset K_b, \sigma \subset K_s$ for some $b, s \in \mathbb{N}$. Set P_{z_β} as the respective isomorphism $i_1 : f_b^{-1}(\sigma_\beta) \to f_s^{-1}(\beta)$ as in the last part of Definition 4.1, viewed as a simplicial embedding of $f_b^{-1}(\sigma_\beta)$ into $f_s^{-1}(\sigma)$. It is again easy to see that this satisfies conditions 1 and 2 of Definition 2.4.

5. It remains to define a $\mathcal{D}$-labelling $\Lambda_\mathcal{R}$ of $\mathcal{R}$. Notice that in our setting $\coprod_{\sigma\in\Sigma} P_\sigma$ is a family of sets of the form $f_n^{-1}(\sigma)$ for various n and σ. The labelling $\Lambda_\mathcal{R}$ is defined analogously to what we have done in point *3*. For every simplex $\tau \in S\left(\coprod_{\sigma\in\Sigma} P_\sigma\right)$ set $\lambda_\mathcal{R}(\tau)$ to be the representative of the type of τ in the family $\mathcal{D}$. Further set $u_\tau^\mathcal{R}$ to be the unique isomorphism $i_0 : \tau \to \lambda_\mathcal{R}(\tau)$ as in the last part of Definition 4.1. It is straightforward to check that the conditions of Definition 3.1 are satisfied.

Thus we have described a constructive Markov system

$$\mathcal{I} = \mathcal{I}(X_0, \mathcal{D}, \mathcal{R}, \Lambda, \Lambda_\mathcal{R}) = \big(\{X_i : i \geq 0\}, \{\Pi_i : i \geq 0\}\big). \tag{28}$$

An easy inductive argument shows that for every n the $(\Lambda_n, \mathcal{D}, \mathcal{R})$-quotient X_{n+1} can be canonically identified with K_{n+2} and every $(\Lambda_n, \mathcal{D}, \mathcal{R})$-map $\Pi_n\colon X_{n+1} \to X_n$ in the above system can be canonically identified with f_{n+1}, so the inverse systems $\mathcal{I}$ and $(K_i, f_i)_{i\geq 1}$ are isomorphic. We skip the details for this argument. □

4.2 A step towards the equivalence of definitions

In this last subsection of the paper we make the following observations. In the previous subsection we showed that the class of spaces which can be described as constructive Markov compacta may be broader than the analogous class related to finitistic Markov compacta. A natural question is whether the two classes coincide. This question obviously reduces to the following one: given a constructive Markov system $\mathcal{I}$, does there exist a semi-barycentric Markov system which is isomorphic to $\mathcal{I}$ (as an inverse system), or a subsequence of $\mathcal{I}$ and which satisfies the distinct types property?

It is not hard to see that a constructive Markov system $\mathcal{I} = \mathcal{I}(X_0, \mathcal{D}, \mathcal{R}, \Lambda, \Lambda_\mathcal{R})$ in the sense of Definition 3.6 is a semi-barycentric Markov system in the sense of Definition 4.8, if we interpret the *type* of a simplex as its label from the family $\mathcal{D}$. We will refer to types associated in this way as the *natural* types for a constructive Markov system. In this subsection we aim to show that under some additional condition on $\mathcal{D}$ we can associate with $\mathcal{I}$ another constructive Markov system, which, after associating natural types as above, satisfies the distinct types property, and which, as an inverse system, is isomorphic to a subsequence of $\mathcal{I}$, see Proposition 4.14 below.

Definition 4.12. Let $\mathcal{D}$ be a good family of simplices. We say that $\mathcal{D}$ has the *distinct label property* if for any simplex $\sigma \in \Sigma$ and any proper faces $\beta, \gamma \subset \sigma$ if $\beta \neq \gamma$ then $\sigma_\beta \neq \sigma_\gamma$.

Example 4.13. Consider the good families from Examples 2.2 and 2.3. Clearly the good family from Example 2.3 has the distinct label property, while the good family from Example 2.2 fails to have it.

Our main result in this subsection is the following.

Proposition 4.14. *Let $\mathcal{I} = \mathcal{I}(X_0, \mathcal{D}, \mathcal{R}, \Lambda, \Lambda_\mathcal{R})$ be a constructive Markov system, where the family $\mathcal{D}$ has the distinct label property. Then there exists another constructive Markov system $\mathcal{I}^+$ such that $\mathcal{I}^+$, with its natural types, has the distinct types property, and the limits of $\mathcal{I}$ and $\mathcal{I}^+$ are homeomorphic. Actually, $\mathcal{I}^+$ can be chosen to coincide, as an inverse system, with a subsequence of $\mathcal{I}$.*

The proof of this proposition requires some preparation. We begin by describing a property of a labelling of a family of rules of replacement and, in the next comment, the importance of it in the reasoning leading to the result above.

Definition 4.15. Let $\mathcal{D}$ be a good family of simplices, and let $\mathcal{R}$ be a good family of rules of replacement for $\mathcal{D}$. We say that a $\mathcal{D}$-labelling $\Lambda_\mathcal{R}$ of the family $\mathcal{R}$ has the *distinct label property* if for every simplex $\sigma \in \Sigma$ the map $\lambda_{\mathcal{R}\restriction P_\sigma}$ is injective.

Fact 4.16. *Let $\mathcal{I} = \mathcal{I}(X_0, \mathcal{D}, \mathcal{R}, \Lambda, \Lambda_\mathcal{R})$ be a constructive Markov system. If the $\mathcal{D}$-labelling $\Lambda_\mathcal{R}$ of the family $\mathcal{R}$ has the distinct label property, then $\mathcal{I}$, with its natural types, has the distinct types property in the sense of Definition 4.4.*

Proof. For every simplex σ in every complex of $\mathcal{I}$ set its type as its label $\lambda(\sigma)$ given by the labelling with the family $\mathcal{D}$. From Lemma 3.3 we see that for each $i \geq 0$, the quotient map from the assembly system over X_i to the $(\Lambda, \mathcal{D}, \mathcal{R})$-quotient is type-preserving. Thus it is enough to see that for every $\sigma \in \Sigma$, simplices in P_σ from the family $\mathcal{R}$ can be assigned pairwise distinct types. This is exactly the condition that $\mathcal{R}$ has the distinct label property. $\square$

We now show how to describe the initial data for a constructive Markov system $\mathcal{I}^+$ as required in the assertion of Proposition 4.14. In short, the idea is to ignore the first term of $\mathcal{I}$ and treat the second term as the base level. This way we provide "enough" labels/types to meet the distinct types property of the Markov system. It turns out that in this approach the distinct label property of the family $\mathcal{D}$ used in the construction of $\mathcal{I}$ is crucial in describing some labelling of $\mathcal{I}^+$, which has the distinct label property.

Definition 4.17. Let $\mathcal{D}$ be a good family of simplices with the distinct label property and let $\mathcal{R}$ be a good family of rules of replacement for $\mathcal{D}$. The *extended family* $\mathcal{D}^+_\mathcal{R} = [\Sigma^+, \{z^+_\beta\}]$ *associated with* $\mathcal{R}$ is defined as follows: Consider $\mathcal{S} = S(\bigcup_{\sigma \in \Sigma} P_\sigma)$. Let $\sim$ be the smallest equivalence relation on $\mathcal{S}$ generated by the relation $\approx$:

$$\tau \approx \rho \iff \tau \subset P_{\sigma_\beta}, \rho \subset P_\sigma \text{ and } P_{z_\beta}(\tau) = \rho \tag{29}$$

for some $\sigma, \sigma_\beta \in \Sigma$. Let Σ^+ consist of one representative from each equivalence class of $\sim$. The embeddings $\{z_\beta^+\}$ are of the following form: pick a simplex $\sigma \in \Sigma^+$ and a face $\beta \subset \sigma$. Then there is a representative $\beta_\sim$ of β with respect to the relation $\sim$. We have two cases:

- if $\beta_\sim = \beta$, set $\sigma_\beta = \beta$ and z_β^+ as the natural inclusion;
- if $\beta_\sim \neq \beta$, then, by definition of $\sim$, there is a sequence of inclusions $\iota_1, \dots, \iota_n$ such that $\iota_1 \circ \ldots \circ \iota_n(\beta_\sim) = \beta$, where for $i = 1, \dots, n$ we have $\iota_i = P^{\pm}_{z_{\beta_i} \restriction \sigma_i}$ for some face β_i, $\sigma_1 := \beta_\sim$ and for $i > 1$ we set $\sigma_i = \iota_1 \circ \ldots \circ \iota_{i-1}(\beta_\sim)$. In this case set $\sigma_\beta = \beta_\sim$ and $z_\beta^+ = \iota_1 \circ \ldots \circ \iota_n$.

Lemma 4.18. *Let $\mathcal{D}$ be a good family of simplices with the distinct label property, $\mathcal{R}$ be a good family of rules of replacement for $\mathcal{D}$ and $\mathcal{D}^+_{\mathcal{R}}$ be the extended family associated with $\mathcal{R}$. Then $\mathcal{D}^+_{\mathcal{R}}$ is a good family of simplices.*

Proof. Since the family $\mathcal{D}$ has the distinct label property, the maps z_β^+ are unique. It is clear that the family $\{z_\beta^+\}$ is closed under composition. $\square$

Definition 4.19. Let $\mathcal{D}$ be a good family of simplices with the distinct label property, $\mathcal{R}$ be a good family of rules of replacement for $\mathcal{D}$, $\mathcal{D}^+_{\mathcal{R}}$ be the extended family (with respect to $\mathcal{R}$), and let $\Lambda_{\mathcal{R}}$ be a $\mathcal{D}$-labelling of $\mathcal{R}$. The *extended family*

$$\mathcal{R}^+ = \left[\left\{ (P_\sigma^+, \pi_\sigma^+) \colon \sigma \in \Sigma^+ \right\}, \left\{ P^+_{z_\beta^+} \right\} \right] \tag{30}$$

of rules of replacement for $\mathcal{D}^+_{\mathcal{R}}$ is defined as follows. Let $\sigma \in \Sigma^+$. Then there is a label $\lambda(\sigma) \in \mathcal{D}$ from the labelling $\Lambda_{\mathcal{R}}$ and a rule of replacement $\lambda(\sigma) \xleftarrow{\pi_{\lambda(\sigma)}} P_{\lambda(\sigma)}$. Set $P_\sigma^+ = P_{\lambda(\sigma)}$ and $\pi_\sigma^+ = \pi_{\lambda(\sigma)}$. Define the rules analogously for all $\sigma \in \Sigma^+$. The family $\{P^+_{z_\beta^+}\}$ of bonding maps has the following form: pick a simplex $\sigma \in \Sigma^+$ and a face $\beta \subset \sigma$. Then there are labels $\lambda(\sigma), \lambda(\sigma_\beta) \in \mathcal{D}$ and an embedding z_β such that $z_\beta(\lambda(\sigma_\beta)) = \lambda(\sigma)_{u_\sigma(\beta)}$. Set $P^+_{z_\beta^+}$ as P_{z_β}.

Fact 4.20. *Let $\mathcal{D}, \mathcal{R}, \mathcal{D}^+_{\mathcal{R}}, \Lambda_{\mathcal{R}}, \mathcal{R}^+$ be as in the above definition. Then the extended family $\mathcal{R}^+$ is a good family of rules of replacement.*

Definition 4.21. Let $\mathcal{D}$, $\mathcal{R}$, $\mathcal{D}^+_{\mathcal{R}}$, $\mathcal{R}^+$, $\Lambda_{\mathcal{R}}$ be as in Definition 4.19. We define the *$\mathcal{D}^+_{\mathcal{R}}$-labelling*

$$\Lambda^+_{\mathcal{R}} = \left[\lambda^+_{\mathcal{R}^+}, \{u_\tau^+\} \right] \tag{31}$$

of the family $\mathcal{R}^+$ in the following way. Pick a complex $P_\tau^+ \in \{P_\sigma : \sigma \in \Sigma^+\}$. Then $P_\tau^+ = P_\sigma$ for some $\sigma \in \Sigma$. Thus every simplex $\varsigma \in P_\tau^+$ has its representative $\varsigma_\sim \in \mathcal{D}^+_{\mathcal{R}}$. We have two cases:

- if $\varsigma_\sim = \varsigma$, set $\lambda^+_{\mathcal{R}^+}(\varsigma) = \varsigma$ and u_ς^+ as the identity;
- if $\varsigma_\sim \neq \varsigma$, then, there is a sequence $\iota_1, \dots, \iota_n$ as in Definition 4.17 such that $\iota_1 \circ \ldots \circ \iota_n(\varsigma) = \varsigma_\sim$. Set $\lambda^+_{\mathcal{R}^+}(\varsigma) = \varsigma_\sim$ and $u_\varsigma^+ = \iota_1 \circ \ldots \circ \iota_n$.

Extend this definition to all simplices in $S(\bigcup_{\sigma\in\Sigma^+} P^+_\sigma)$. It remains to check the two conditions of Definition 3.1. Pick simplices $\sigma_\beta, \sigma \in \Sigma^+$ where $\beta \subset \sigma$ is a proper face. Then for any two simplices $\tau_1 \in P^+_{\sigma_\beta}, \tau_2 \in P^+_\sigma$ such that $P^+_{z^+_\beta}(\tau_1) = \tau_2$ we have that $\tau_1 \sim \tau_2$, so from the definition we have $\lambda^+_{\mathcal{R}^+}(\tau_1) = \tau_{1\sim} = \tau_{2\sim} = \lambda^+_{\mathcal{R}^+}(\tau_2)$. Condition (22) is also easily met.

Fact 4.22. *Let $\mathcal{D}$, $\mathcal{R}$, $\mathcal{D}^+_\mathcal{R}$, $\mathcal{R}^+$, $\Lambda_\mathcal{R}, \Lambda^+_\mathcal{R}$ be as in the above definition. The $\mathcal{D}^+_\mathcal{R}$-labelling $\Lambda^+_\mathcal{R}$ has the distinct label property.*

The proof of this fact will become immediate if we give a characterization of the relation $\sim$, which is the subject of the following.

Lemma 4.23. *Let $\mathcal{S} = S(\coprod_{\sigma\in\Sigma} P_\sigma)$ and let $\sim$ be the smallest equivalence relation on $\mathcal{S}$ generated by the relation (29). Then for any $\sigma_1, \sigma_2 \in \mathcal{S}$ such that $\sigma_1 \subset P_\tau, \sigma_2 \subset P_\rho$ we have*

$$\begin{aligned} \sigma_1 \sim \sigma_2 \iff \sigma_1 = \sigma_2 \textit{ or } \sigma_1 \approx \sigma_2 \textit{ or } \sigma_2 \approx \sigma_1 \textit{ or } \sigma_1 \subset \pi_\tau^{-1}(\beta), \sigma_2 \subset \pi_\rho^{-1}(\gamma) \textit{ and} \\ \exists \sigma \in \Sigma\, \exists \overline{\sigma} \subset P_\sigma \quad \sigma = \tau_\beta, \sigma = \rho_\gamma, P_{z_\beta}(\overline{\sigma}) = \sigma_1, P_{z_\gamma}(\overline{\sigma}) = \sigma_2 \end{aligned} \tag{32}$$

where β and γ are proper faces of the simplices τ and ρ respectively.

Proof. (of Lemma 4.23)
The proof is very similar to the proof of Lemma 1.3 and we skip it. □

Proof. (of Fact 4.22)
Pick a complex $P^+_\tau \in \{P_\sigma : \sigma \in \Sigma^+\}$. Then $P^+_\tau = P_\tau$ for some $\tau \in \Sigma$. Suppose that there are simplices $\rho_1, \rho_2 \in P^+_\tau$ such that $\rho_1 \neq \rho_2$ but $\lambda^+_{\mathcal{R}^+}(\rho_1) = \lambda^+_{\mathcal{R}^+}(\rho_2)$. Then $\rho_1 \sim \rho_2$. Since $\rho_1 \neq \rho_2$, we have the situation as in the fourth term in the disjunction (32). But this contradicts the fact that $\mathcal{D}$ is a family with the distinct type property. □

Finally we can prove the main result of this subsection, Proposition 4.14, in the following way.

Proof. (of Proposition 4.14)
Consider the system $\mathcal{I}^+ = \mathcal{I}(X^+_0, \mathcal{D}^+_\mathcal{R}, \mathcal{R}^+, \Lambda^+, \Lambda^+_\mathcal{R})$, where

- $X^+_0 = X_1 \in \mathcal{I}$;
- $\mathcal{D}^+_\mathcal{R}$ is the extended family associated with $\mathcal{R}$;
- $\mathcal{R}^+$ is the extended family of rules of replacement for $\mathcal{D}^+_\mathcal{R}$;
- Λ^+ has the following description. Notice that there is an obvious $\mathcal{D}^+_\mathcal{R}$-labelling of $\mathcal{R}$ – it is defined in the same way as the labelling presented in Definition 4.21, since $\bigcup_{\sigma\in\Sigma^+} P^+_\sigma = \bigcup_{\sigma\in\Sigma} P_\sigma$. We then apply Remark 3.4 to $\mathcal{D}_a = \mathcal{D}, \mathcal{D}_b = \mathcal{D}^+_\mathcal{R}$ and obtain a unique $\mathcal{D}^+_\mathcal{R}$-labelling of the $(\Lambda, \mathcal{D}, \mathcal{R})$-quotient $X_1 = X^+_0$, which we call Λ^+;
- $\Lambda^+_\mathcal{R}$ is the $\mathcal{D}^+_\mathcal{R}$-labelling of $\mathcal{R}^+$ as in Definition 4.21;

We will show that $\mathcal{I}^+$ can be canonically identified, as an inverse system, with the inverse system $(X_i, f_i)_{i\geq 1}$. This will conclude the proof, as $\mathcal{I}^+$, with its natural types, has the distinct types property – this follows from Fact 4.22 and Fact 4.16.

Suppose we have already constructed the complex $X_1 \in \mathcal{I}$. Similarly to the framework presented in Section 2.4 we consider two assembly systems over X_1: one built using the data from $\mathcal{I}$, the other one using the data from $\mathcal{I}^+$. Notice that in this case we have

$$\coprod_{\tau\subset X_1} P_{\lambda(\tau)} = \coprod_{\tau\subset X_1} P_{\lambda^+(\tau)}. \tag{33}$$

We proceed by showing that the embeddings $i_{\rho\tau}$ defined using the family $\mathcal{R}$ generate the same equivalence relation on $\coprod_{\tau\subset X_1} P_{\lambda(\tau)}$ as those defined using the family $\mathcal{R}^+$. This is immediate, since for both constructions we use the same bonding maps. Indeed, let $\rho, \tau \subset X_1$ be such that $\rho \subset \tau$, and let $\lambda(\rho), \lambda(\sigma), \lambda^+(\rho), \lambda^+(\sigma)$ be their labels with respect to the labellings Λ_1, Λ^+. We have the commutative diagrams

$$\begin{tikzcd}
\rho \arrow[r, "u_\rho"'] \arrow[d, hook] & \lambda(\rho) \arrow[d, dashed, "z_{u_\tau(\rho)}"] & P_{\lambda(\rho)} \arrow[l] \arrow[d, "P_{z_{u_\tau(\rho)}}"] \\
\tau \arrow[r, "u_\tau"] & \lambda(\tau) & P_{\lambda(\tau)} \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\rho \arrow[r, "u^+_\rho"'] \arrow[d, hook] & \lambda^+(\rho) \arrow[d, "z^+_{u^+_\tau(\rho)}"] & P^+_{\lambda^+(\rho)} \arrow[l] \arrow[d, "P^+_{z^+_{u^+_\tau(\rho)}}"] \\
\tau \arrow[r, "u^+_\tau"] & \lambda^+(\tau) & P^+_{\lambda^+(\tau)} \arrow[l]
\end{tikzcd} \tag{34}$$

By definition we have that $P^+_{\lambda^+(\rho)} = P_{\lambda(\rho)}$, $P^+_{\lambda^+(\tau)} = P_{\lambda(\tau)}$ and $P^+_{z^+_{u^+_\tau(\rho)}} = P_{z_{u_\tau(\rho)}}$. This way we see that the $(\Lambda_1, \mathcal{D}, \mathcal{R})$-quotient over X_1 can be canonically identified with the $(\Lambda^+, \mathcal{D}^+_{\mathcal{R}}, \mathcal{R}^+)$-quotient. Moreover, by definition of the replacement maps for the family $\mathcal{R}^+$, the $(\Lambda_1, \mathcal{D}, \mathcal{R})$-map and the $(\Lambda^+, \mathcal{D}^+_{\mathcal{R}}, \mathcal{R}^+)$-map coincide. The statement of the theorem now follows from an easy inductive argument. $\square$

In this last subsection we relied on an additional property assumed for a good family of simplices – the distinct label property. A further line of research can be dedicated to determining whether for any constructive Markov system $\mathcal{I} = \mathcal{I}(X_0, \mathcal{D}, \mathcal{R}, \Lambda, \Lambda_{\mathcal{R}})$ (where $\mathcal{D}$ need not have the distinct label property), there exists another constructive Markov system $\mathcal{I}'$ such that $\mathcal{I}'$ has the distinct types property (as a Markov system) and the limits of $\mathcal{I}$ and $\mathcal{I}'$ are homeomorphic. This way we would achieve full equivalence of the definition of constructive Markov compactum as presented in this paper with the definition of a finitistic Markov compactum.

5 Example - reflection trees of graphs as constructive Markov compacta

In this section we aim to express certain topological spaces, called *reflection trees of graphs* (as described in Section 2 of [Świ19]), as Markov compacta. We adopt the conventions used in Section 2 of [Świ19]. In particular, by a *graph* (or *topological* *graph)* we mean the underlying topological space $X = |\Gamma|$ of a finite simplicial graph Γ. The natural cell structure of a graph X is the coarsest cell structure on X. The vertices of this structure (also called *the essential vertices of* X) are these points $x \in X$ which locally split X into different than 2 number of connected components.

5.1 Good family of simplices for a reflection tree of graphs

We begin by describing a good family of simplices associated to a finite simplicial graph Γ. Denote by $S^0(\Gamma)$ the set of vertices of Γ, and by $S^1(\Gamma)$ the set of edges of Γ.

Definition 5.1. The *good family of simplices* $\mathcal{D} = \mathcal{D}(\Gamma)$ *for a graph* Γ is a family

$$\mathcal{D}(\Gamma) = \big[\{o\} \cup S^0(\Gamma) \cup \bigcup_{e \in S^1(\Gamma)} V(e), \{z_\beta\}_{\beta \in B}\big] \tag{35}$$

where o is some extra 0-simplex and where for each 1-simplex e of Γ, the set $V(e)$ consists of four copies of e, and we write $V(e) = \{e_1, e_2, e_3, e_4\}$. The family $\{z_\beta\}_{\beta \in B}$ of embeddings has the following description. Let e be a 1-simplex. The proper faces of copies $e_i = [v_i, w_i]$ of $e = [v, w]$ in $V(e)$ are equipped with the following inclusions:

- $(e_1)_{v_1} = v, (e_1)_{w_1} = w$, and set $z_{v_1} : (e_1)_{v_1} \to v_1, z_{w_1} : (e_1)_{w_1} \to w_1$ to be the maps $v \mapsto v_1, w \mapsto w_1$ respectively;

- $(e_2)_{v_2} = v, (e_2)_{w_2} = o$ and set $z_{v_2} : (e_2)_{v_2} \to v_2$ to be the map $v \mapsto v_2$ and z_{w_2} to be the map $o \mapsto w_2$;

- $(e_3)_{v_3} = o, (e_3)_{w_3} = w$ and set z_{v_3} to be the map $o \mapsto v_3$ and $z_{w_3} : (e_3)_{w_3} \to w_3$ to be the map $w \mapsto w_3$;

- $(e_4)_{v_4} = o, (e_4)_{w_4} = o$ and set z_{v_4}, z_{w_4} to be the maps $o \mapsto v_4, o \mapsto w_4$ respectively.

Fact 5.2. *The family* $\mathcal{D}(\Gamma)$ *as above is a good family of simplices.*

5.2 Rules of replacement for $\mathcal{D}(\Gamma)$

We begin by describing certain simplicial complexes that will appear in the description of the rules of replacement for $\mathcal{D}(\Gamma)$.

Definition 5.3. Let Γ be a simplicial graph, let $X = |\Gamma|$ and let x be a point in X. The *blow-up of* Γ *at* x, denoted by $\Gamma^{\#}(x)$ is the simplicial graph obtained in the following way. Attach to $X \setminus \{x\}$ as many points as the number of components into which x locally splits X (which will become vertices of valence 1 in $\Gamma^{\#}(x)$). Denote the set of these attached vertices by $\mathcal{P}_x$. To describe the simplicial structure, we consider two cases:

- *x is a vertex of* Γ. Denote by $[v_1, x], [v_2, x], \ldots, [v_n, x]$ the edges of Γ adjacent to x. In the blow-up they are replaced with n distinct edges $[v_1, x_{v_1}], [v_2, x_{v_2}], \ldots, [v_n, x_{v_n}]$. The simplicial structure on the remaining part of Γ remains unchanged;

- *x is an interior point of an edge* $[v, w]$. In this case $[v, w]$ splits into two edges $[v, x_v]$ and $[x_w, w]$. The simplicial structure on the remaining part of Γ remains unchanged.

For any finite subset $J \subset X$, denote by $\Gamma^{\#}(J)$ the simplicial graph obtained by performing blow-ups at all points $x \in J$ (the result does not depend on the order).

Remark 5.4. Topologically $\Gamma^{\#}(x)$ coincides with $X^{\#}(x)$ as described in [Świ19], page 4. In other words, the geometric realization $|\Gamma^{\#}(x)|$ of $\Gamma^{\#}(x)$ is $X^{\#}(x)$.

Definition 5.5. Let $\mathcal{D}(\Gamma)$ be the good family of simplices from Definition 5.1. The family $\mathcal{R} = \mathcal{R}_{\mathcal{D}(\Gamma)}$ *of rules of replacement for* $\mathcal{D}(\Gamma)$ is a family of rules of replacement with the following description:

- for the 0-simplex o set $P_o = a, \pi_a \colon a \mapsto o$, where a is some 0-simplex;
- for any other 0-simplex v set $P_v = \Gamma^{\#}(v), \pi_v \colon P_v \mapsto v$;

and for any 1-simplex $e = [v, w]$, the copies of e in $V(e)$ are equipped with the following rules. Let $B_i = [v_i, b_{e_i}^{v_i}] \sqcup [b_{e_i}^{w_i}, w_i]$ where $1 \leq i \leq 4$, be the disjoint union of 1-simplices obtained by performing a blow-up of e_i at its barycenter.

- for the edge e_1 set
$$P_{e_1} = {}^{B_1 \sqcup \Gamma^{\#}(v) \sqcup \Gamma^{\#}(w) \sqcup \Gamma^{\#}(b_e)}\diagup_{\sim} \tag{36}$$
where $\sim$ is the equivalence relation induced by the following equivalences: identify the vertex v_1 with the vertex v_w in the edge $[w, v_w]$ of $\Gamma^{\#}(v)$; the vertex w_1 with the vertex w_v in the edge $[v, w_v]$ of $\Gamma^{\#}(w)$; the vertex $b_{e_1}^{v_1}$ with the vertex $(b_e)_v$ in the edge $[v, (b_e)_v]$ of $\Gamma^{\#}(b_e)$ and finally the vertex $b_{e_1}^{w_1}$ with the vertex $(b_e)_w$ in the edge $[w, (b_e)_w]$ of $\Gamma^{\#}(b_e)$. The map $\pi_{e_1} \colon P_{e_1} \to {e_1}'$ is the unique simplicial map with the following properties:
$$\begin{cases} \pi_{e_1 \restriction \Gamma^{\#}(v)} \colon \Gamma^{\#}(v) \mapsto v_1 \\ \pi_{e_1 \restriction \Gamma^{\#}(w)} \colon \Gamma^{\#}(w) \mapsto w_1 \\ \pi_{e_1 \restriction \Gamma^{\#}(b_e)} \colon \Gamma^{\#}(b_e) \mapsto b_{e_1} \\ \pi_{e_1 \restriction B_1} = \varphi_1 \end{cases}$$
where $\varphi_1 \colon B_1 \to {e_1}'$ is the simplicial map taking $v_1 \in B_1$ to $v_1 \in {e_1}'$, $w_1 \in B_1$ to $w_1 \in {e_1}'$ and both $b_{e_1}^{w_1}, b_{e_1}^{v_1}$ to b_{e_1}.
- for the edge e_2 set
$$P_{e_2} = {}^{B_2 \sqcup \Gamma^{\#}(v) \sqcup \Gamma^{\#}(b_e)}\diagup_{\sim} \tag{37}$$
where $\sim$ is defined similarly as above. The map $\pi_{e_2} \colon P_{e_2} \to {e_2}'$ is the unique simplicial map with the following properties:
$$\begin{cases} \pi_{e_2 \restriction \Gamma^{\#}(v)} \colon \Gamma^{\#}(v) \mapsto v_2 \\ \pi_{e_2 \restriction \Gamma^{\#}(b_{e_2})} \colon \Gamma^{\#}(b_{e_2}) \mapsto b_{e_2} \\ \pi_{e_2 \restriction B_2} = \varphi_2 \end{cases}$$

where $\varphi_2\colon B_2 \to {e_2}'$ is the simplicial map taking $v_2 \in B_2$ to $v_2 \in {e_2}'$, $w_2 \in B_2$ to $w_2 \in {e_2}'$ and both $b_{e_2}^{w_2}, b_{e_2}^{v_2}$ to b_{e_2}.

- for the edge e_3 set
$$P_{e_3} = {}^{B_3 \sqcup \Gamma^{\#}(w) \sqcup \Gamma^{\#}(b_e)}\big/_{\sim} \tag{38}$$
where $\sim$ is defined similarly as above. The map $\pi_{e_3}\colon P_{e_3} \to {e_3}'$ is the unique simplicial map with the following properties:
$$\begin{cases} \pi_{e_3 \restriction \Gamma^{\#}(w)}\colon \Gamma^{\#}(w) \mapsto w_3 \\ \pi_{e_3 \restriction \Gamma^{\#}(b_e)}\colon \Gamma^{\#}(b_e) \mapsto b_{e_3} \\ \pi_{e_3 \restriction B_3} = \varphi_3 \end{cases}$$
where $\varphi_3\colon B_3 \to {e_3}'$ is the simplicial map taking $v_3 \in B_3$ to $v_3 \in {e_3}'$, $w_3 \in B_3$ to $w_3 \in {e_3}'$ and both $b_{e_3}^{w_3}, b_{e_3}^{v_3}$ to b_{e_3}.

- for the edge e_4 set
$$P_{e_4} = {}^{B_4 \sqcup \Gamma^{\#}(b_e)}\big/_{\sim} \tag{39}$$
where $\sim$ is defined similarly as above. The map $\pi_{e_4}\colon P_{e_4} \to {e_4}'$ is the unique simplicial map with the following properties:
$$\begin{cases} \pi_{e_4 \restriction \Gamma^{\#}(b_e)}\colon \Gamma^{\#}(b_e) \mapsto b_{e_4} \\ \pi_{e_4 \restriction B_4} = \varphi_4 \end{cases}$$
where $\varphi_4\colon B_4 \to {e_4}'$ is the simplicial map taking $v_4 \in B_4$ to $v_4 \in {e_4}'$, $w_4 \in B_4$ to $w_4 \in {e_4}'$ and both $b_{e_4}^{w_4}, b_{e_4}^{v_4}$ to b_{e_4}.

The bonding maps have the following description (we present the bonding maps only for e_2, and the rest of them is defined in an analogous way):

- for $z_{v_2} : (e_2)_{v_2} \to v$ set $P_{z_{v_2}} : P_{(e_2)_{v_2}} \to P_{e_2}$ to be the inclusion of $\Gamma^{\#}(v)$ to the copy of $\Gamma^{\#}(v)$ in P_{e_2};
- for $z_{w_2}\colon o \mapsto w$ set $P_{z_{w_2}} : P_{(e_2)_{w_2}} \to P_{e_2}$ as the map taking a to w_2.

Fact 5.6. *The family $\mathcal{R} = \mathcal{R}_{\mathcal{D}(\Gamma)}$ as above is a good family of rules of replacement.*

5.3 Labelling of $\mathcal{R}_{\mathcal{D}(\Gamma)}$ with the family $\mathcal{D}(\Gamma)$

Definition 5.7. Let $\mathcal{R}$ be a good family of rules of replacement as in Definition 5.5. The $\mathcal{D}(\Gamma)$- *labelling* $\Lambda_{\mathcal{R}_{\mathcal{D}(\Gamma)}}$ *of* $\mathcal{R}_{\mathcal{D}(\Gamma)}$ is defined in the following way. Let P_σ be one of the summand complexes of $\coprod_{\tau\in\mathcal{D}} P_\tau$. We carry out the labelling in the following way:

1. if $P_\sigma = a$, which happens when $\sigma = o$ set
$$\lambda(a) = o,\ u_a = a \mapsto o \tag{40}$$

2. if $P_\sigma = \Gamma^\#(v)$ for some vertex v, which happens when $\sigma = v$ we have the following rules. Let $e \in P_\sigma$ be an edge. If e is of the form $[w_i, v_{w_i}]$ (so e was adjacent to the vertex at which the blow-up was made), set $\lambda(e) = e_3$ and u_e to be the simplicial map taking w_i to w and v_{w_i} to v. Moreover set $\lambda(w_i) = w, u_{w_i} = id_w$ and $\lambda(v_{w_i}) = o, u_{v_{w_i}} = v_{w_i} \mapsto o$. For all other edges e set $\lambda(e) = e_1$, $u_e = id_{e_1}$.

3. if $P_\sigma = P_{e_i}$ for some edge e of Γ, and some $i \in \{1, 2, 3, 4\}$, then we have the following rules. Let us assume $i = 2$ (cases $i = 1, 3, 4$ are done analogously). Let us denote $e_2 = [v_2, w_2]$. For the 1-simplices $\tau_1 = [v_2, b^{v_2}_{e_2}], \tau_2 = [b^{w_2}_{e_2}, w_2]$ in B_2 set
$$\lambda(\tau_1) = e_4, \lambda(\tau_2) = e_4, \tag{41}$$
and further set u_{τ_1} to be the isomoprhism induced by extending linearly the map $v_2 \mapsto v_4, b^{v_2}_\sigma \mapsto w_4$ to the map $\tau_1 \mapsto e_4$, and analogously set u_{τ_2} to be the isomoprhism induced by extending linearly the map $w_2 \mapsto w_4, b^{w_2}_\sigma \mapsto v_4$ to the map $\tau_2 \mapsto e_4$. For the simplices $\tau_3 = [v, b^v_e], \tau_4 = [b^w_e, w]$ in $\Gamma^\#(b_e)$ set
$$\lambda(\tau_3) = e_2, \lambda(\tau_4) = e_3, \tag{42}$$
and further set u_{τ_3} to be the isomoprhism induced by extending linearly the map $v \mapsto v_2, b^v_e \mapsto w_2$ to the map $\tau_1 \mapsto e_2$, and analogously set u_{τ_2} to be the isomoprhism induced by extending linearly the map $w \mapsto w_3, b^w_e \mapsto v_3$ to the map $\tau_2 \mapsto e_3$. The labelling of $\Gamma^\#(v)$ is done in the same way that what is done in Point 2 above. For all other edges e set $\lambda(e) = e_1$, $u_e = id_{e_1}$.

5.4 Construction

In this subsection we give a description of a specific subsequence S_0 in the reflection system $\mathcal{S}_X$ (as denoted in the comment below Lemma 2.2 of [Świ19]) for some graph X and show that it can be identified with the constructive Markov system
$$\big(\{X_i : i \geq 0\}, \{\Pi_i : i \geq 0\} = \mathcal{I}(\Gamma, \mathcal{D}(\Gamma), \mathcal{R}_{\mathcal{D}(\Gamma)}, \Lambda_\Gamma, \Lambda_{\mathcal{R}_{\mathcal{D}(\Gamma)}}\big), \tag{43}$$
where Γ is such that $X = |\Gamma|$, and $\mathcal{D}(\Gamma), \mathcal{R}_{\mathcal{D}(\Gamma)}, \Lambda_{\mathcal{R}_{\mathcal{D}(\Gamma)}}$ are described in sections $5.1 - 5.3$ and Λ_Γ is the tautological labelling of Γ as in Example 2.9. We begin by describing a certain dense subset of X. Consider the set
$$D_b = \bigcup_{i=1}^{\infty} B_i, \tag{44}$$
where the sets $B_i, i > 1$ are the sets of vertices of the i-th barycentric subdivision of Γ. We also consider the partition of D_b into the sets $D_i = B_i \setminus \bigcup_{j<i} B_j$, each of which is obviously finite.

Fact 5.8. *The set D_b is a countable dense subset of X containing all essential vertices of X.*

Let us recall some of the notation from [Świ19], Section 2, subsection titled *"The spaces X_F"*. Let T be a tree each vertex of which has valence equal to the cardinality of the subset D_b. Denote by V_T, E_T the sets of vertices and (unoriented) edges of T, respectively. To each vertex $t \in V_T$ associate a copy of the graph X, and denote it X_t. Equip also T with a labelling $\lambda : E(T) \to D_b$ such that for any $t \in V_T$, denoting by $A_t T$ the set of edges in T adjacent to t, the restriction of λ to $A_t T$ is a bijection on D_b. Such a labelling clearly exists, and is unique up to an automorphism of T. Let $\mathcal{F}$ be the poset of all finite subtrees of T, ordered by inclusion.

Consider the chain $\{F_n\}_{n \geq 0}$ in $\mathcal{F}$, defined recursively as follows. Pick any $t_0 \in V_T$.

- Set F_0 to be the subtree of T coinciding with the single vertex t_0;
- The tree F_{n+1} is the unique subtree of T satisfying the following 3 conditions:
  1. $F_n \subset F_{n+1}$;
  2. any vertex of F_{n+1} is adjacent to some vertex of F_n;
  3. for any vertex $t \in F_n$ (so that $t \in F_k \setminus \bigcup_{j<k} F_j$ for some $k \leq n$), the closed star of t in F_{n+1} consists of edges $e \in A_t T$ such that $\lambda(e) \in \bigcup_{j \leq n+1-k} D_j$, and vertices adjacent to those edges.

Observe that the chain $\{F_n\}_{n \geq 0}$ is cofinal in $\mathcal{F}$.

We now recall the notion of the *standard reflection system for a graph* X, which is an inverse system of the form

$$S(X, D_b) = \big(\{X_F : F \in \mathcal{F}\}, \{\pi_{F',F} : F \subset F'\}\big), \tag{45}$$

where the spaces X_F and the maps $\pi_{F',F} : F \subset F'$ are described in [Świ19], Section 2, subsections titled *"The spaces X_F"* and *"The maps $\pi_{F',F}$"*. Recall also that the *reflection tree of graphs of* X, denoted by $\mathcal{X}^r(X)$, is, by definition, the inverse limit of the system $S(X, D_b)$.

Consider now the subsequence S_0 of $S(X, D_b)$, where

$$S_0 = \big(\{X_{F_n}\}_{n \geq 0}, \{\pi_{F_{n+1},F_n}\}_{n \geq 0}\big), \tag{46}$$

and where the sequence $\{F_n\}$ is described above. Recall that the spaces X_{F_n} have the following description:

$$X_{F_n} = \bigsqcup_{t \in V_{F_n}} \Gamma_t^{\#}(\lambda(A_t F_n)) \big/ \sim \tag{47}$$

where λ is the labelling described below Fact 5.8 and the relation $\sim$ is induced by the following equivalences: for each edge $e = [t_1, t_2] \in E_{F_n}$ and each $p \in \mathcal{P}_{\lambda(e)}$, identify $p \in \mathcal{P}_{\lambda(e)} \subset \Gamma_{t_1}^{\#}(\lambda(A_{t_1} F_n))$ with $p \in \mathcal{P}_{\lambda(e)} \subset \Gamma_{t_2}^{\#}(\lambda(A_{t_2} F_n))$.

Proposition 5.9. *The system S_0 can be canonically identified with the constructive Markov system (43).*

Proof. We proceed by showing that both systems consist of the same spaces and the same maps. Indeed, let $X_n \in \mathcal{I}$. We note that X_n is, in our setting, the $(\Lambda, \mathcal{D}, \mathcal{R})$-quotient of the set

$$\bigsqcup_{\sigma \in S(X_{n-1})} Y_\sigma \tag{48}$$

where $Y_\sigma = P_{\lambda(\sigma)}$ and λ is the labelling from the system (45) (for each σ there is a seperate copy of the complex $P_{\lambda(\sigma)}$). Upon inspection of the definitions of the spaces Y_σ and the spaces $\Gamma_t^{\#}(\lambda(A_t F_n))$ apprearing in (47) we see that this $(\Lambda, \mathcal{D}, \mathcal{R})$-quotient is precisely X_{F_n}.

Finally, it is an easy observation that the maps $\{\pi_{F_{i+1},F_i} : F_i \subset F_{i+1}\}$ in S_0 coincide with the maps $\{\Pi_i : i \geq 0\}$ in $\mathcal{I}$. □

Corollary 5.10. Every reflection tree of graphs, as described in Definition 2.3 in [Świ19], is a constructive Markov compactum.

Proof. In the above Proposition, we proved that every reflection inverse system can be canonically identified with some constructive Markov system, and furthermore, since $\{F_n\}_{n\geq 0}$ is cofinal in $\mathcal{F}$, the inverse limit of S_0 is $\mathcal{X}^r(X)$. In other words, we see that

$$\mathcal{X}^r(X) = \varprojlim S(X, D_b) = \varprojlim S_0 = \varprojlim \mathcal{I}(\Gamma, \mathcal{D}(\Gamma), \mathcal{R}_{\mathcal{D}(\Gamma)}, \Lambda_\Gamma, \Lambda_{\mathcal{R}_{\mathcal{D}(\Gamma)}}), \tag{49}$$

which yields the Corollary. □

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