Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
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Maciej Kucharski

Problem Waringa i metoda łuków Hardy'ego-Littlewooda

Praca licencjacka
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dr. hab. Mariusza Mirka

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University of Wrocław
Faculty of Mathematics and Computer Science
Mathematical Institute
Joint Studies in Computer Science and Mathematics

Maciej Kucharski

Waring's problem and the Hardy-Littlewood circle method

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dr. hab. Mariusz Mirek

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Contents

1. Waring's problem

Waring's problem asks whether for each natural number k there exists an integer s such that any natural number is the sum of at most s kth powers. Since every number can be represented as the sum of ones, this is equivalent to the question if there exists s such that the equation

$$x_1^k + \cdots + x_s^k = N \tag{1}$$

has any solutions in integers for all sufficiently large integers N.

Let $r_{k,s}(N)$ be the number of solutions of equation (1). We will follow the method of Hardy, Littlewood and Ramanujan described in [1] to obtain the estimate of $r_{k,s}$. Let A be a set of non-negative integers and let $f(z) = \sum_{a\in A} z^a$. Then

$$f(z)^s = \sum_{n=0}^{\infty} r_{A,s}(n) z^n,$$

where $r_{A,s}(n)$ is the number of representations of n as the sum of s elements of A. Since elements of A are non-negative, if we want to recover $r_{A,s}(n)$, we can truncate this series to get the polynomial $p(z) = \sum_{\substack{a\in A\\ a\leqslant N}}$. Then

$$p(z)^s = \sum_{m=0}^{sN} r_{A,s}^{(N)}(m) z^m,$$

where $r_{A,s}^{(N)}(m)$ is the number of representations of m as the sum of s elements of A not exceeding N. For $m \leqslant N$ we have $r_{A,s}^{(N)}(m) = r_{A,s}(m)$. If we let $z = e(\alpha) = e^{2\pi i\alpha}$, we get

$$F(\alpha) = p(e(\alpha)) = \sum_{\substack{a\in A\\ a\leqslant N}} e(a\alpha)$$

and

$$F(\alpha)^s = \sum_{m=0}^{sN} r_{A,s}^{(N)}(m) e(m\alpha).$$

Since

$$\int_0^1 e(m\alpha)e(-n\alpha)d\alpha = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases},$$

we have

$$r_{A,s}(N) = \int_0^1 F(\alpha)^s e(-N\alpha)d\alpha.$$

If we want to apply this to Waring's problem, we let A be the set of kth powers and $P = [N^{\frac{1}{k}}]$. Then

$$F(\alpha) = \sum_{\substack{a \in A \\ a \leqslant N}} e(\alpha a) = \sum_{n=1}^{P} e(\alpha n^k)$$

and

$$r_{k,s}(N) = r_{A,s}(N) = \int_0^1 F(\alpha)^s e(-\alpha N) d\alpha.$$

Our aim is to estimate this integral.

2. Useful lemmas

First we establish some tools needed in the circle method.

Lemma 1. *Let f be a continuously differentiable function and let $U(t) = \sum_{1 \leqslant n \leqslant t} u(n)$. Let a and b be non-negative integers with $a < b$. Then*

$$\sum_{n=a+1}^{b} u(n) f(n) = U(b) f(b) - U(a) f(a) - \int_a^b U(t) f'(t) dt.$$

Proof. First observe that

$$f(n+1) - f(n) = \int_n^{n+1} f'(t) dt$$

and

$$U(n)(f(n+1) - f(n)) = \int_n^{n+1} U(t) f'(t) dt.$$

Therefore

$$\begin{aligned}\sum_{n=a+1}^{b} u(n)f(n) &= \sum_{n=a+1}^{b} (U(n) - U(n-1))f(n) \\ &= \sum_{n=a+1}^{b} U(n)f(n) - \sum_{n=a}^{b-1} U(n)f(n+1) \\ &= U(b)f(b) - U(a)f(a) - \sum_{n=a}^{b-1} U(n)(f(n+1) - f(n)) \\ &= U(b)f(b) - U(a)f(a) - \sum_{n=a}^{b-1} \int_{n}^{n+1} U(t)f'(t)dt \\ &= U(b)f(b) - U(a)f(a) - \int_{a}^{b} U(t)f'(t)dt.\end{aligned}$$

□

Lemma 2. *Let*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

be the gamma function.
1. If $x > 0$, then $\Gamma(x) \geqslant \frac{1}{e}$.
2. If $x \in [1, 2]$, then $\Gamma(x) \leqslant 1$.

Proof.
1.
If $x \in (0, 1)$, then we have

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \geqslant \int_0^1 t^{x-1}e^{-t}dt \geqslant \frac{1}{e}.$$

If $x \geqslant 1$, then

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \geqslant \int_1^\infty t^{x-1}e^{-t}dt \geqslant \int_1^\infty e^{-t}dt = \frac{1}{e}.$$

2. Assume that $x > 0$ and compute the second derivative of Γ:

$$\Gamma''(x) = \frac{d^2}{dx^2}\int_0^\infty t^{x-1}e^{-t}dt = \int_0^\infty \frac{\partial^2}{\partial x^2} t^{x-1}e^{-t}dt = \int_0^\infty t^{x-1}\log^2(t)e^{-t}dt > 0.$$

Thus, Γ is convex for $x > 0$. Noting that $\Gamma(1) = \Gamma(2) = 1$, we get the desired result. □

Theorem 3 (Dirichlet's theorem)**.** *Let α and $Q \geqslant 1$ be real numbers. Then there exist integers a and q such that $1 \leqslant q \leqslant Q$, $(a, q) = 1$ and*

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{qQ} \leqslant \frac{1}{q^2}.$$

Proof. Let $N = [Q]$. Suppose that $\{q\alpha\} \in [0, \frac{1}{N+1})$ for some positive integer $q \leqslant N$. Taking $a = [q\alpha]$, we get

$$0 \leqslant \{q\alpha\} = q\alpha - [q\alpha] = q\alpha - a < \frac{1}{N+1}$$

and

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q(N+1)} < \frac{1}{qQ} \leqslant \frac{1}{q^2}.$$

Similarly, if $\{q\alpha\} \in [\frac{N}{N+1}, 1)$ for some positive integer $q \leqslant N$ and if $a = [q\alpha] + 1$, then

$$\frac{N}{N+1} \leqslant \{q\alpha\} = q\alpha - a + 1 < 1.$$

This implies that

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q(N+1)} < \frac{1}{qQ} \leqslant \frac{1}{q^2}.$$

Now suppose that $\{q\alpha\} \in [\frac{1}{N+1}, \frac{N}{N+1})$ for all $q = 1, \ldots, N$. This means that there are N numbers lying in $N - 1$ intervals $[\frac{i}{N+1}, \frac{i+1}{N+1})$ $(i = 1, \ldots, N-1)$. By the pigeonhole principle, there exist integers $i \in [1, N-1]$ and $1 \leqslant q_1 < q_2 \leqslant N$ such that

$$\{q_1\alpha\},\ \{q_2\alpha\} \in \left[\frac{i}{N+1}, \frac{i+1}{N+1}\right).$$

Let $q = q_2 - q_1 \in [1, N-1]$ and $a = [q_2\alpha] - [q_1\alpha]$. Then

$$|q\alpha - a| = |(q_2\alpha - [q_2\alpha]) - (q_1\alpha - [q_1\alpha])| = |\{q_2\alpha\} - \{q_1\alpha\}| < \frac{1}{N+1} < \frac{1}{Q}.$$

□

Definition 1. $\|\alpha\| = \min(|n - \alpha| : n \in \mathbb{Z}) = \min(\{\alpha\}, \{1 - \alpha\})$

Observation.
1. $\|\alpha + \beta\| \leqslant \|\alpha\| + \|\beta\|$ for all real numbers α and β.
2. $|\sin \pi\alpha| = \sin \pi \|\alpha\|$ for all real numbers α.

Fact 4. *If $0 < \alpha < \frac{1}{2}$, then $2\alpha < \sin \pi\alpha < \pi\alpha$.*

Definition 2. $e(t) = e^{2\pi i t}$

Lemma 5. *For every real number α and all integers $N_1 < N_2$*

$$\left|\sum_{n=N_1+1}^{N_2} e(\alpha n)\right| \leqslant \min\left(N_2 - N_1, \frac{1}{2\|\alpha\|}\right) \leqslant \min\left(N_2 - N_1, \frac{1}{\|\alpha\|}\right).$$

Proof. Since $|e(\alpha n)| \leqslant 1$, we have

$$\left|\sum_{n=N_1+1}^{N_2} e(\alpha n)\right| \leqslant N_2 - N_1.$$

If $\alpha \notin \mathbb{Z}$, then $\|\alpha\| > 0$ and $e(\alpha) \neq 1$. We have

$$\begin{aligned}
\left|\sum_{n=N_1+1}^{N_2} e(\alpha n)\right| &= \left|e(\alpha(N_1+1)) \sum_{n=0}^{N_2-N_1-1} e(\alpha)^n\right| \\
&= \left|\frac{e(\alpha(N_2 - N_1)) - 1}{e(\alpha) - 1}\right| \leqslant \frac{2}{|e(\alpha) - 1|} \\
&= \frac{2}{\left|e(\frac{\alpha}{2}) - e(\frac{-\alpha}{2})\right|} = \frac{2}{|2i \sin \pi\alpha|} \\
&= \frac{1}{|\sin \pi\alpha|} = \frac{1}{\sin \pi \|\alpha\|} \leqslant \frac{1}{2\|\alpha\|} \leqslant \frac{1}{\|\alpha\|}
\end{aligned}$$

□

Lemma 6. *Let α be a real number and let q and a be integers such that $q \geqslant 1$ and $(a, q) = 1$. If $\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q^2}$, then*

$$\sum_{1 \leqslant r \leqslant \frac{q}{2}} \frac{1}{\|\alpha r\|} \leqslant 6q \log q$$

Proof. The lemma holds for $q = 1$, so we can assume that $q \geqslant 2$. For each integer r there exist integers $s(r) \in [0, \frac{q}{2}]$ and $m(r)$ such that

$$\frac{s(r)}{q} = \left\|\frac{ar}{q}\right\| = \pm\left(\frac{ar}{q} - m(r)\right).$$

Since $(a, q) = 1$, it follows that $s(r) = 0$ if and only if $r \equiv 0 \pmod{q}$ and therefore $s(r) \in [1, \frac{q}{2}]$ if $r \in [1, \frac{q}{2}]$. Let $\alpha - \frac{a}{q} = \frac{\theta}{q^2}$, where $-1 \leqslant \theta \leqslant 1$. Then

$$\alpha r = \frac{ar}{q} + \frac{\theta r}{q^2} = \frac{ar}{q} + \frac{\theta'}{2q},$$

where

$$|\theta'| = \left|\frac{2\theta r}{q}\right| \leqslant |\theta| \leqslant 1.$$

From the triangle inequality we have

$$\begin{aligned}
\|\alpha r\| &= \left\|\frac{ar}{q} + \frac{\theta'}{2q}\right\| \\
&= \left\|m(r) \pm \frac{s(r)}{q} + \frac{\theta'}{2q}\right\| \\
&= \left\|\frac{s(r)}{q} \pm \frac{\theta'}{2q}\right\| \\
&\geqslant \left\|\frac{s(r)}{q}\right\| - \left\|\frac{\theta'}{2q}\right\| \\
&\geqslant \frac{s(r)}{q} - \frac{1}{2q}.
\end{aligned}$$

Let $1 \leqslant r_1 \leqslant r_2 \leqslant \frac{q}{2}$. We will show that $s(r_1) = s(r_2)$ if and only if $r_1 = r_2$. If

$$\frac{s(r_1)}{q} = \frac{s(r_2)}{q},$$

then

$$\pm\left(\frac{ar_1}{q} - m(r_1)\right) = \pm\left(\frac{ar_2}{q} - m(r_2)\right)$$

and

$$ar_1 \equiv \pm ar_2 \pmod{q}.$$

Since $(a, q) = 1$, we have $r_1 \equiv \pm r_2 \pmod{q}$ and $r_1 = r_2$ as $1 \leqslant r_1 \leqslant r_2 \leqslant \frac{q}{2}$. Thus

$$\left\{\left\|\frac{ar}{q}\right\| : 1 \leqslant r \leqslant \frac{q}{2}\right\} = \left\{\frac{s(r)}{q} : 1 \leqslant r \leqslant \frac{q}{2}\right\} = \left\{\frac{s}{q} : 1 \leqslant s \leqslant \frac{q}{2}\right\}.$$

and

$$\begin{aligned}
\sum_{1 \leqslant r \leqslant \frac{q}{2}} \frac{1}{\|\alpha r\|} &\leqslant \sum_{1 \leqslant r \leqslant \frac{q}{2}} \frac{1}{\frac{s(r)}{q} - \frac{1}{2q}} \\
&= 2q \sum_{1 \leqslant s \leqslant \frac{q}{2}} \frac{1}{2s - 1} \\
&\leqslant 2q \sum_{1 \leqslant s \leqslant \frac{q}{2}} \frac{1}{s} \leqslant 2q(1 + \log \tfrac{q}{2}) \\
&\leqslant 2q(1 + \log q) \leqslant 6q \log q.
\end{aligned}$$

□

Lemma 7. *Let α be a real number. If $\left|\alpha - \frac{a}{q}\right|$, where $q \geqslant 1$ and $(a,q) = 1$, then for any $V \geqslant 0$ and natural number h*

$$\sum_{r=1}^{q} \min\left(V, \frac{1}{\|\alpha(hq+r)\|}\right) \leqslant 8V + 24q\log q.$$

Proof. Let $\alpha = \frac{a}{q} + \frac{\theta}{q^2}$ for some $-1 \leqslant \theta \leqslant 1$. Then

$$\begin{aligned}\alpha(hq+r) &= ah + \frac{ar}{q} + \frac{\theta h}{q} + \frac{\theta r}{q^2}\\ &= ah + \frac{ar}{q} + \frac{[\theta h] + \{\theta h\}}{q} + \frac{\theta r}{q^2}\\ &= ah + \frac{ar + [\theta h] + \delta(r)}{q},\end{aligned}$$

where

$$-1 \leqslant \delta(r) = \{\theta h\} + \frac{\theta r}{q} < 2.$$

For $r = 1, \ldots, q$ let r' be an integer such that $\{\alpha(hq+r)\} = \frac{ar+[\theta h]+\delta(r)}{q} - r'$. Let $0 \leqslant t \leqslant 1 - \frac{1}{q}$. If

$$t \leqslant \{\alpha(hq+r)\} \leqslant t + \frac{1}{q},$$

then

$$qt \leqslant ar - qr' + [\theta h] + \delta(r) \leqslant qt + 1.$$

It follows that

$$ar - qr' \leqslant qt - [\theta h] + 1 - \delta(r) \leqslant qt - [\theta h] + 2$$

and

$$ar - qr' \geqslant qt - [\theta h] - \delta(r) > qt - [\theta h] - 2$$

Thus, $ar - qr'$ is in an interval containing exactly four distinct integers. If $1 \leqslant r_1 \leqslant r_2 \leqslant q$ and $ar_1 - qr_1' = ar_2 - qr_2'$, then $ar_1 \equiv ar_2 \pmod q$. Since $(a,q) = 1$, $r_1 \equiv r_2 \pmod q$ and $r_1 = r_2$. It follows that for any $t \in [0, 1 - \frac{1}{q}]$ there are at most four integers $r \in [1,q]$ such that

$$\{\alpha(hq+r)\} \in [t, t + \tfrac{1}{q}]$$

Observe that

$$\|\alpha(hq+r)\| \in [t, t + \tfrac{1}{q}]$$

if and only if

$$\{\alpha(hq+r)\} \in [t, t+\tfrac{1}{q}] \quad \text{or} \quad 1-\{\alpha(hq+r)\} \in [t, t+\tfrac{1}{q}].$$

The second relation is equivalent to

$$\{\alpha(hq+r)\} \in [t', t'+\tfrac{1}{q}]$$

for $0 \leqslant t' = 1-\frac{1}{q}-t \leqslant 1-\frac{1}{q}$. It follows that for any $t \in [0, 1-\frac{1}{q}]$ there are at most eight integers $r \in [1,q]$ such that $\|\alpha(hq+r)\| \in [t, t+\frac{1}{q}]$.
For $s = 0, 1, \ldots$ let $I(s) = [\frac{s}{q}, \frac{s+1}{q}]$. Let us estimate the sum

$$\sum_{r=1}^{q} \min\left(V, \frac{1}{\|\alpha(hq+r)\|}\right).$$

If $\|\alpha(hq+r)\| \in I(0)$, then we can use the fact that $\min\left(V, \frac{1}{\|\alpha(hq+r)\|}\right) \leqslant V$.
If $\|\alpha(hq+r)\| \in I(s)$ for some $s \geqslant 1$, we have $\min\left(V, \frac{1}{\|\alpha(hq+r)\|}\right) \leqslant \frac{1}{\|\alpha(hq+r)\|} \leqslant \frac{q}{s}$.
Since $\|\alpha(hq+r)\| \in I(s)$ for some $s < \frac{q}{2}$, we have

$$\sum_{r=1}^{q} \min\left(V, \frac{1}{\|\alpha(hq+r)\|}\right) \leqslant 8V + 8 \sum_{1 \leqslant s < \frac{q}{2}} \frac{q}{s} \leqslant 8V + 24q\log q.$$

□

Lemma 8. *Let α be a real number. If $\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q^2}$, where $q \geqslant 1$ and $(a,q) = 1$, then for any $U \geqslant 1$ and natural number n*

$$\sum_{1 \leqslant k \leqslant U} \min\left(\frac{n}{k}, \frac{1}{\|\alpha k\|}\right) \leqslant \left(\frac{32n}{q} + 24U + 30q\right) \log 4qU.$$

Proof. Let $k = hq + r$, where $1 \leqslant r \leqslant q$ and $0 \leqslant h < \frac{U}{q}$. Then

$$S = \sum_{1 \leqslant k \leqslant U} \min\left(\frac{n}{k}, \frac{1}{\|\alpha k\|}\right) \leqslant \sum_{0 \leqslant h < \frac{U}{q}} \sum_{r=1}^{q} \min\left(\frac{n}{hq+r}, \frac{1}{\|\alpha(hq+r)\|}\right).$$

If $h = 0$ and $1 \leqslant r \leqslant \frac{q}{2}$, then by Lemma 6 we have

$$\sum_{1 \leqslant r \leqslant \frac{q}{2}} \min\left(\frac{n}{r}, \frac{1}{\|\alpha r\|}\right) \leqslant \sum_{1 \leqslant r \leqslant \frac{q}{2}} \frac{1}{\|\alpha r\|} \leqslant 6q\log q.$$

Otherwise $\frac{1}{hq+r} < \frac{2}{(h+1)q}$ and thus

$$S \leqslant 6q \log q + \sum_{0 \leqslant h < \frac{U}{q}} \sum_{r=1}^{q} \min\left(\frac{2n}{(h+1)q}, \frac{1}{\|\alpha(hq+r)\|}\right)$$

Observe that

$$\sum_{0 \leqslant h < \frac{U}{q}} \frac{1}{h+1} \leqslant 1 + \log\left(\frac{U}{q} + 1\right) \leqslant 2\log\left(\frac{U}{q} + 2\right) \leqslant 2\log(U + 2q) \leqslant 2\log 4Uq.$$

Let $V = \frac{2n}{(h+1)q}$. Then by Lemma 7

$$\begin{aligned} S &\leqslant 6q \log q + \sum_{0 \leqslant h < \frac{U}{q}} \sum_{r=1}^{q} \min\left(\frac{2n}{(h+1)q}, \frac{1}{\|\alpha(hq+r)\|}\right) \\ &\leqslant 6q \log q + \sum_{0 \leqslant h < \frac{U}{q}} \left(\frac{16n}{(h+1)q} + 24q \log q\right) \\ &\leqslant 6q \log q + \frac{16n}{q} \sum_{0 \leqslant h < \frac{U}{q}} \frac{1}{h+1} + 24\left(\frac{U}{q} + 1\right) q \log q \\ &\leqslant 6q \log q + \frac{32n}{q} \log 4qU + 24U \log q + 24q \log q \\ &\leqslant \left(\frac{32n}{q} + 24U + 30q\right) \log 4qU. \end{aligned}$$

□

Lemma 9. *Let α be a real number. If $\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q^2}$, where $q \geqslant 1$ and $(a, q) = 1$, then for any real numbers U and n we have*

$$\sum_{1 \leqslant k \leqslant U} \min\left(n, \frac{1}{\|\alpha k\|}\right) \leqslant \left(30q + 24U + 8n + \frac{8Un}{q}\right) \max(1, \log q)$$

Proof. We proceed as in the previous proof.

$$\begin{aligned}
&\sum_{1\leqslant k\leqslant U} \min\left(n, \frac{1}{\|\alpha k\|}\right) \\
&\leqslant \sum_{0\leqslant h<\frac{U}{q}} \sum_{r=1}^{q} \min\left(n, \frac{1}{\|\alpha(hq+r)\|}\right) \\
&\leqslant 6q\log q + \sum_{0\leqslant h<\frac{U}{q}} (8n + 24q\log q) \\
&\leqslant 6q\log q + \left(\frac{U}{q}+1\right)(8n + 24q\log q) \\
&= 30q\log q + 24U\log q + \frac{8Un}{q} + 8n \\
&\leqslant \left(30q + 24U + 8n + \frac{8Un}{q}\right)\max(1, \log q).
\end{aligned}$$

□

Definition 3.

$$\Delta_d(f)(x) = f(x+d) - f(x)$$

$$\Delta_{d_l,\dots,d_1} = \Delta_{d_l} \circ \Delta_{d_{l-1}} \circ \cdots \circ \Delta_{d_1}$$

Lemma 10. *Let N_1, N_2, N be integers such that $N_1 < N_2$ and $0 \leqslant N_2 - N_1 \leqslant N$. Let f be a real-valued function and*

$$S(f) = \sum_{n=N_1+1}^{N_2} e(f(n)).$$

Then

$$|S(f)|^2 = \sum_{|d|<N} S_d(f),$$

where

$$S_d(f) = \sum_{n\in I(d)} e(\Delta_d(f)(n)) \quad \text{and} \quad I(d) = [N_1+1-d, N_2-d] \cap [N_1+1, N_2].$$

Proof.

$$\begin{aligned}
|S(f)|^2 &= S(f)\overline{S(f)} \\
&= \sum_{m=N_1+1}^{N_2} e(f(m)) \sum_{n=N_1+1}^{N_2} \overline{e(f(n))} \\
&= \sum_{n=N_1+1}^{N_2} \sum_{m=N_1+1}^{N_2} e(f(m)-f(n)) \\
&= \sum_{n=N_1+1}^{N_2} \sum_{d=N_1+1-n}^{N_2-n} e(f(n+d)-f(n)) \\
&= \sum_{n=N_1+1}^{N_2} \sum_{d=N_1+1-n}^{N_2-n} e(\Delta_d(f)(n))
\end{aligned}$$

Note that

$$\begin{cases} N_1+1 \leqslant n \leqslant N_2 \\ N_1+1-n \leqslant d \leqslant N_2-n \end{cases} \quad \Leftrightarrow \quad \begin{cases} N_1+1 \leqslant n \leqslant N_2 \\ N_1+1-d \leqslant n \leqslant N_2-d \\ -(N_2-N_1-1) \leqslant d \leqslant N_2-N_1-1 \end{cases}$$

and $N_2-N_1-1 < N$. Therefore

$$\begin{aligned}
|S(f)|^2 &= \sum_{d=-(N_2-N_1-1)}^{N_2-N_1-1} \sum_{n\in I(d)} e(\Delta_d(f)(n)) \\
&= \sum_{|d|<N} \sum_{n\in I(d)} e(\Delta_d(f)(n)) \\
&= \sum_{|d|<N} S_d(f).
\end{aligned}$$

□

Lemma 11. *Let N_1, N_2, N, l be integers such that $l \geqslant 1$, $N_1 < N_2$ and $0 \leqslant N_2 - N_1 \leqslant N$. Let f be a real-valued function and*

$$S(f) = \sum_{n=N_1+1}^{N_2} e(f(n)).$$

Then

$$|S(f)|^{2^l} \leqslant (2N)^{2^l-l-1} \sum_{|d_1|<N} \cdots \sum_{|d_l|<N} S_{d_l,\dots,d_1}(f),$$

where

$$S_{d_l,\dots,d_1}(f) = \sum_{n\in I(d_l,\dots,d_1)} e(\Delta_{d_l,\dots,d_1}(f)(n))$$

and $I(d_l,\dots,d_1)$ is some interval of consecutive integers contained in $[N_1+1,N_2]$.

Proof. By induction on l. The case $l=1$ has been proven in the previous lemma. Assume that it is true for some $l\geqslant 1$. Using the inductive hypothesis and the Cauchy-Schwarz inequality we get

$$\begin{aligned}
|S(f)|^{2^{l+1}} &= \left(|S(f)|^{2^l}\right)^2 \\
&\leqslant \left((2N)^{2^l-l-1}\sum_{|d_1|<N}\cdots\sum_{|d_l|<N}|S_{d_l,\dots,d_1}(f)|\right)^2 \\
&= (2N)^{2^{l+1}-2l-2}\left(\sum_{|d_1|<N}\cdots\sum_{|d_l|<N}|S_{d_l,\dots,d_1}(f)|\right)^2 \\
&\leqslant (2N)^{2^{l+1}-2l-2}(2N)^l\sum_{|d_1|<N}\cdots\sum_{|d_l|<N}|S_{d_l,\dots,d_1}(f)|^2
\end{aligned}$$

By the previous lemma there is an interval

$$I(d_{l+1},d_l,\dots,d_1)\subseteq I(d_l,\dots,d_1)\subseteq [N_1+1,N_2]$$

such that

$$|S_{d_l,\dots,d_1}(f)|^2 = \sum_{|d_{l+1}|<N} S_{d_{l+1},d_l,\dots,d_1}(f),$$

and thus

$$|S(f)|^{2^{l+1}} \leqslant (2N)^{2^{l+1}-(l+1)-1}\sum_{|d_1|<N}\cdots\sum_{|d_l|<N}\sum_{|d_{l+1}|<N} S_{d_{l+1},d_l,\dots,d_1}(f).$$

□

Lemma 12. *Let $k\geqslant 1$ and $1\leqslant l\leqslant k$. Then*

$$\Delta_{d_l,\dots,d_1}(x^k) = \sum_{\substack{j_1+\cdots+j_l+j=k\\ j\geqslant 0,j_1,\dots,j_l\geqslant 1}} \frac{k!}{j!j_1!\cdots j_l!}d_1^{j_1}\cdots d_l^{j_l}x^j = d_1\cdots d_l p_{k-l}(x),$$

where p_{k-l} is a of polynomial degree $k-l$ with leading coefficient $\frac{k!}{(k-l)!}$. If $d_1,\dots,d_l$ are integers, then p_{k-l} has integer coefficients.

Proof. By induction on l. For $l = 1$ we have

$$\Delta_{d_1}(x^k) = (x+d_1)^k - x^k = \sum_{j=0}^{k-1} \binom{k}{j} d_1^{k-j} x^j = \sum_{\substack{j_1+j=k \\ j\geqslant 0, j_1 \geqslant 1}} \frac{k!}{j!j_1!} d_1^{j_1} x^j.$$

Let $1 \leqslant l \leqslant k-1$ and assume that the formula holds for l. Then

$$\begin{aligned}
\Delta_{d_{l+1},d_l,\ldots,d_1}(x^k) &= \Delta_{d_{l+1}}\left(\Delta_{d_l,\ldots,d_1}(x^k)\right) \\
&= \sum_{\substack{j_1+\cdots+j_l+m=k \\ m\geqslant 0, j_1,\ldots,j_l \geqslant 1}} \frac{k!}{m!j_1!\cdots j_l!} d_1^{j_1}\cdots d_l^{j_l} \Delta_{d_{l+1}}(x^m) \\
&= \sum_{\substack{j_1+\cdots+j_l+m=k \\ m, j_1,\ldots,j_l \geqslant 1}} \frac{k!}{m!j_1!\cdots j_l!} d_1^{j_1}\cdots d_l^{j_l} \sum_{\substack{j_{l+1}+j=m \\ j\geqslant 0, j_{l+1} \geqslant 1}} \frac{m!}{j!j_{l+1}!} d_{l+1}^{j_{l+1}} x^j \\
&= \sum_{\substack{j_1+\cdots+j_l+m=k \\ m, j_1,\ldots,j_l \geqslant 1}} \sum_{\substack{j_{l+1}+j=m \\ j\geqslant 0, j_{l+1} \geqslant 1}} \frac{k!}{j!j_1!\cdots j_l!j_{l+1}!} d_1^{j_1}\cdots d_l^{j_l} d_{l+1}^{j_{l+1}} x^j \\
&= \sum_{\substack{j_1+\cdots+j_l+j_{l+1}+j=k \\ j\geqslant 0, j_1,\ldots,j_l,j_{l+1} \geqslant 1}} \frac{k!}{j!j_1!\cdots j_l!j_{l+1}!} d_1^{j_1}\cdots d_l^{j_l} d_{l+1}^{j_{l+1}} x^j.
\end{aligned}$$

Since the multinomial coefficients $\frac{k!}{j!j_1!\cdots j_l!}$ are integers, it follows that p_{k-l} has integer coefficients, provided that $d_1, \ldots, d_l$ are integers. □

Corollary. *Let $f(x) = \alpha x^k + \cdots + \alpha_0$. Then*

$$\Delta_{d_{k-1},\ldots,d_1}(f)(x) = d_1 \cdots d_{k-1} k! \alpha x + \beta.$$

Lemma 13. *Let $1 \leqslant l \leqslant k$. If $|d_1|, \ldots, |d_l|, x \leqslant P$, then $\Delta_{d_l,\ldots,d_1}(x^k) \leqslant (l+1)^k P^k$.*

Proof. By Lemma 12 we have

$$\begin{aligned}
\left|\Delta_{d_l,\ldots,d_1}(x^k)\right| &= \left| \sum_{\substack{j_1+\cdots+j_l+j=k \\ j\geqslant 0, j_1,\ldots,j_l \geqslant 1}} \frac{k!}{j!j_1!\cdots j_l!} d_1^{j_1}\cdots d_l^{j_l} x^j \right| \\
&\leqslant \sum_{\substack{j_1+\cdots+j_l+j=k \\ j\geqslant 0, j_1,\ldots,j_l \geqslant 1}} \frac{k!}{j!j_1!\cdots j_l!} P^{j_1+\cdots+j_l+j} \\
&\leqslant \sum_{\substack{j_1+\cdots+j_l+j=k \\ j, j_1,\ldots,j_l \geqslant 0}} \frac{k!}{j!j_1!\cdots j_l!} P^k \\
&= (l+1)^k P^k.
\end{aligned}$$

□

Lemma 14. *Let $d(n)$ be the number of divisors of n. Then for any $\varepsilon > 0$ $d(n) \leqslant d_\varepsilon n^\varepsilon$, where $d_\varepsilon = \frac{e^{\frac{1}{\varepsilon}} 2^{1+\varepsilon}}{\varepsilon}$.*

Proof. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1, \ldots, p_k$ are primes. Then

$$d(n) = (\alpha_1 + 1) \cdots (\alpha_k + 1)$$

and

$$\frac{d(n)}{n^\varepsilon} = \prod_{j=1}^{k} \frac{\alpha_j + 1}{p_j^{\varepsilon \alpha_j}}$$

Now we can divide factors of the above product into two classes:
1. $p_j \geqslant e^{\frac{1}{\varepsilon}}$. Then $p_j^{\varepsilon\alpha_j} \geqslant e^{\alpha_j} \geqslant 1 + \alpha_j$, so $\frac{\alpha_j+1}{p_j^{\varepsilon\alpha_j}} \leqslant 1$. Thus, the overall contribution of such factors is less then 1.
2. $p_j < e^{\frac{1}{\varepsilon}}$. Let $f(x) = \frac{x+1}{2^{\varepsilon x}}$ for $x \geqslant 0$. Then

$$f'(x) = \frac{1 - \varepsilon(x+1)\log(2)}{2^{\varepsilon x}}$$

Solving $f'(x) = 0$ we get $x = \frac{1}{\varepsilon \log(2)} - 1$. It follows that

$$f\left(\frac{1}{\varepsilon \log(2)} - 1\right) = \frac{2^\varepsilon}{e\varepsilon \log(2)}$$

is the maximum of f and for every $1 \leqslant j \leqslant k$

$$\frac{\alpha_j + 1}{p_j^{\varepsilon\alpha_j}} \leqslant \frac{\alpha_j + 1}{2^{\varepsilon\alpha_j}} \leqslant \frac{2^\varepsilon}{e\varepsilon\log(2)}.$$

If we want to bound that product from above we can neglect factors of class 1 and estimate the number of factors of class 2 by $e^{\frac{1}{\varepsilon}}$. Thus

$$d(n) \leqslant e^{\frac{1}{\varepsilon}} \frac{2^\varepsilon}{e\varepsilon\log(2)} n^\varepsilon \leqslant \frac{e^{\frac{1}{\varepsilon}} 2^\varepsilon}{\varepsilon} n^\varepsilon = \frac{d_\varepsilon}{2} n^\varepsilon.$$

However, we double the constant so that this estimate is still valid if we count both positive and negative divisors. □

Lemma 15. *Let $k \geqslant 1$, $K = 2^{k-1}$ and $\varepsilon \geqslant 0$. Let $f(x) = \alpha x^k + \cdots + \alpha_0$ be a polynomial with real coefficients and*

$$S(f) = \sum_{n=1}^{N} e(f(n)).$$

Then

$$|S(f)|^K \leqslant k(2N)^{K-1} + 2^{K-1}N^{K-k}d^k_{\frac{\varepsilon}{k^2}}(k!N)^\varepsilon \sum_{m=1}^{k!N^{k-1}} \min\left(N, \frac{1}{\|m\alpha\|}\right).$$

Proof. Applying Lemma 11 with $N_1 = 0$, $N_2 = N$, $l = k-1$ we get

$$|S(f)|^K \leqslant (2N)^{K-k} \sum_{|d_1|<N} \cdots \sum_{|d_{k-1}|<N} \left|S_{d_{k-1},\dots,d_1}(f)\right|,$$

where

$$S_{d_{k-1},\dots,d_1}(f) = \sum_{n\in I(d_{k-1},\dots,d_1)} e(\Delta_{d_{k-1},\dots,d_1}(f)(n))$$

and $I(d_{k-1},\dots,d_1) = [N_1+1, N_2] \subseteq [1,N]$. Since $|e(t)| = 1$, we have $\left|S_{d_{k-1},\dots,d_1}(f)\right| \leqslant N$. By Lemma 12

$$\Delta_{d_{k-1},\dots,d_1}(f)(x) = d_1\cdots d_{k-1}k!\alpha x + \beta = \lambda x + \beta.$$

and by Lemma 5

$$\begin{aligned}
\left|S_{d_{k-1},\dots,d_1}(f)\right| &= \left|\sum_{n\in I(d_{k-1},\dots,d_1)} e(\Delta_{d_{k-1},\dots,d_1}(f)(n))\right| \\
&= \left|\sum_{n=N_1+1}^{N_2} e(\lambda n + \beta)\right| \\
&= \left|\sum_{n=N_1+1}^{N_2} e(\lambda n)\right| \\
&\leqslant \frac{1}{\|\lambda\|} = \frac{1}{\|d_1\cdots d_{k-1}k!\alpha\|},
\end{aligned}$$

so we have

$$\left|S_{d_{k-1},\dots,d_1}(f)\right| \leqslant \min\left(N, \frac{1}{\|d_1\cdots d_{k-1}k!\alpha\|}\right).$$

Therefore

$$\begin{aligned}
|S(f)|^K &\leqslant (2N)^{K-k} \sum_{|d_1|<N} \cdots \sum_{|d_{k-1}|<N} \left|S_{d_{k-1},\dots,d_1}(f)\right| \\
&\leqslant (2N)^{K-k} \sum_{|d_1|<N} \cdots \sum_{|d_{k-1}|<N} \min\left(N, \frac{1}{\|d_1\cdots d_{k-1}k!\alpha\|}\right).
\end{aligned}$$

If $d_1 \cdots d_{k-1} = 0$, then $\min\left(N, \frac{1}{\|d_1 \cdots d_{k-1} k! \alpha\|}\right) = N$. There are fewer than $(k-1)(2N)^{k-2}$ choices of $d_1, \ldots, d_{k-1}$ such that $d_1 \cdots d_{k-1} = 0$, so

$$\begin{aligned}|S(f)|^K &\leqslant (2N)^{K-k}(k-1)(2N)^{k-2}N \\ &\quad + (2N)^{K-k} \sum_{1 \leqslant |d_1| < N} \cdots \sum_{1 \leqslant |d_{k-1}| < N} \min\left(N, \frac{1}{\|d_1 \cdots d_{k-1} k! \alpha\|}\right) \\ &\leqslant k(2N)^{K-1} + 2^{K-1} N^{K-k} \sum_{1 \leqslant d_1 < N} \cdots \sum_{1 \leqslant d_{k-1} < N} \min\left(N, \frac{1}{\|d_1 \cdots d_{k-1} k! \alpha\|}\right).\end{aligned}$$

Since by Lemma 14 $d(m) \leqslant d_\varepsilon m^\varepsilon$, it follows that the number of choices of $d_1, \ldots, d_{k-1}$ such that $m = d_1 \cdots d_{k-1} k!$ is at most $d(m)^{k-1} \leqslant (d_\varepsilon m^\varepsilon)^{k-1}$. In our case $1 \leqslant d_1 \cdots d_{k-1} k! \leqslant k! N^{k-1}$, so

$$d(m)^{k-1} \leqslant (d_\varepsilon m^\varepsilon)^{k-1} \leqslant d_\varepsilon^k m^{\varepsilon k} = d_{\frac{\varepsilon}{k}}^k m^\varepsilon \leqslant d_{\frac{\varepsilon}{k}}^k (k! N^k)^\varepsilon = d_{\frac{\varepsilon}{k^2}}^k k!^{\frac{\varepsilon}{k}} N^\varepsilon \leqslant d_{\frac{\varepsilon}{k^2}}^k (k! N)^\varepsilon.$$

Therefore

$$\begin{aligned}|S(f)|^K &\leqslant k(2N)^{K-1} + 2^{K-1} N^{K-k} \sum_{1 \leqslant d_1 \leqslant N} \cdots \sum_{1 \leqslant d_{k-1} \leqslant N} \min\left(N, \frac{1}{\|d_1 \cdots d_{k-1} k! \alpha\|}\right) \\ &\leqslant k(2N)^{K-1} + 2^{K-1} N^{K-k} d_{\frac{\varepsilon}{k^2}}^k (k! N)^\varepsilon \sum_{m=1}^{k! N^{k-1}} \min\left(N, \frac{1}{\|m\alpha\|}\right).\end{aligned}$$

□

2.1. Weyl's inequality

Theorem 16 (Weyl's inequality). *For $k \geqslant 2$ let $f(x) = \alpha x^k + \cdots + \alpha_0$ be a polynomial with real coefficients and suppose that there exist integers a and q such that $q \geqslant 1$, $(a, q) = 1$ and $\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q^2}$. Let $K = 2^{k-1}$, $\varepsilon > 0$ and*

$$S(f) = \sum_{n=1}^{N} e(f(n)).$$

Then

$$|S(f)| \leqslant 2N^{1+\varepsilon} \left(d_{\frac{\varepsilon}{2k^2 K}}^k k!^\varepsilon \frac{2kK}{\varepsilon}\right)^{\frac{1}{K}} \left(\frac{30q}{N^k} + \frac{32k!}{N} + \frac{8k!}{q}\right)^{\frac{1}{K}}.$$

Proof. Since $|S(f)| \leqslant N$, the result follows if $q \geqslant N^k$. Thus, we will assume that $1 \leqslant q < N^k$. Then $\log q \leqslant k \log N \leqslant \frac{k}{\varepsilon} N^\varepsilon$. By Lemma 15 we have

$$|S(f)|^K \leqslant k(2N)^{K-1} + 2^{K-1} N^{K-k} d_{\frac{\varepsilon}{k^2}}^k (k! N)^\varepsilon \sum_{m=1}^{k! N^{k-1}} \min\left(N, \frac{1}{\|m\alpha\|}\right).$$

By Lemma 9 we have

$$\sum_{m=1}^{k!N^{k-1}} \min\left(N, \frac{1}{\|m\alpha\|}\right) \leqslant \left(30q + 24k!N^{k-1} + 8N + \frac{8k!N^k}{q}\right)\max(1, \log q)$$
$$\leqslant \left(30q + 32k!N^{k-1} + \frac{8k!N^k}{q}\right)\frac{k}{\varepsilon}N^{\varepsilon}$$
$$= \frac{k}{\varepsilon}N^{k+\varepsilon}\left(\frac{30q}{N^k} + \frac{32k!}{N} + \frac{8k!}{q}\right).$$

Therefore

$$\begin{aligned}|S(f)|^K &\leqslant k(2N)^{K-1} + 2^{K-1}N^{K-k}d^k_{\frac{\varepsilon}{k^2}}(k!N)^{\varepsilon}\sum_{m=1}^{k!N^{k-1}} \min\left(N, \frac{1}{\|m\alpha\|}\right) \\ &\leqslant k(2N)^{K-1} + 2^{K-1}N^{K-k}d^k_{\frac{\varepsilon}{k^2}}(k!N)^{\varepsilon}\frac{k}{\varepsilon}N^{k+\varepsilon}\left(\frac{30q}{N^k} + \frac{32k!}{N} + \frac{8k!}{q}\right) \\ &\leqslant 2^K N^{K+2\varepsilon} d^k_{\frac{\varepsilon}{k^2}} k!^{\varepsilon}\frac{k}{\varepsilon}\left(\frac{30q}{N^k} + \frac{32k!}{N} + \frac{8k!}{q}\right) \\ &= 2^K N^{K+\varepsilon} d^k_{\frac{\varepsilon}{2k^2}} k!^{\frac{\varepsilon}{2}}\frac{2k}{\varepsilon}\left(\frac{30q}{N^k} + \frac{32k!}{N} + \frac{8k!}{q}\right)\end{aligned}$$

Thus, taking Kth root and replacing ε with $\frac{\varepsilon}{K}$, we get

$$\begin{aligned}|S(f)| &\leqslant 2N^{1+\varepsilon}\left(d^k_{\frac{\varepsilon}{2k^2K}} k!^{\frac{\varepsilon}{2K}}\frac{2kK}{\varepsilon}\right)^{\frac{1}{K}}\left(\frac{30q}{N^k} + \frac{32k!}{N} + \frac{8k!}{q}\right)^{\frac{1}{K}} \\ &\leqslant 2N^{1+\varepsilon}\left(d^k_{\frac{\varepsilon}{2k^2K}} k!^{\varepsilon}\frac{2kK}{\varepsilon}\right)^{\frac{1}{K}}\left(\frac{30q}{N^k} + \frac{32k!}{N} + \frac{8k!}{q}\right)^{\frac{1}{K}}.\end{aligned}$$

□

The next two theorems are applications of Weyl's inequality:

Theorem 17. *Let $k \geqslant 2$ and let $\frac{a}{q}$ be a rational number with $q \geqslant 1$ and $(a, q) = 1$. Then*

$$|S(q,a)| = \left|\sum_{x=1}^{q} e\left(\frac{ax^k}{q}\right)\right| \leqslant 2\left(d^k_{\frac{\varepsilon}{2k^2K}} 60k!^{1+\varepsilon}\frac{2kK}{\varepsilon}\right)^{\frac{1}{K}} q^{1-\frac{1}{K}+\varepsilon}.$$

Proof. Let $f(x) = \frac{ax^k}{q}$, $N = q$ and apply Weyl's inequality:

$$\begin{aligned}|S(q,a)| &\leqslant 2q^{1+\varepsilon}\left(d^k_{\frac{\varepsilon}{2k^2K}} k!^{\varepsilon}\frac{2kK}{\varepsilon}\right)^{\frac{1}{K}}\left(\frac{30}{q^{k-1}} + \frac{32k!}{q} + \frac{8k!}{q}\right)^{\frac{1}{K}} \\ &\leqslant 2\left(d^k_{\frac{\varepsilon}{2k^2K}} 60k!^{1+\varepsilon}\frac{2kK}{\varepsilon}\right)^{\frac{1}{K}} q^{1-\frac{1}{K}+\varepsilon}.\end{aligned}$$

□

Theorem 18. *Let $k \geqslant 2$, $N \geqslant 2$ and let $\frac{a}{q}$ be a rational number with $q \geqslant 1$, $(a,q)=1$ and $N^{\frac{1}{2}} \leqslant q \leqslant N^{k-\frac{1}{2}}$. Then there exists $\delta > 0$ such that*

$$\left|\sum_{n=1}^{N} e\left(\frac{an^k}{q}\right)\right| \leqslant 2\left(d^k_{\frac{1-2\delta K}{4k^2K^2}} 60k!^{1+\frac{1}{2K}-\delta}\frac{4kK^2}{1-2\delta K}\right)^{\frac{1}{K}} N^{1-\delta}.$$

Proof. Apply Weyl's inequality with $f(x) = \frac{ax^k}{q}$:

$$\begin{aligned}
|S(f)| &\leqslant 2N^{1+\varepsilon}\left(d^k_{\frac{\varepsilon}{2k^2K}} k!^{\varepsilon}\frac{2kK}{\varepsilon}\right)^{\frac{1}{K}}\left(\frac{30q}{N^k}+\frac{32k!}{N}+\frac{8k!}{q}\right)^{\frac{1}{K}}\\
&\leqslant 2N^{1+\varepsilon}\left(d^k_{\frac{\varepsilon}{2k^2K}} k!^{\varepsilon}\frac{2kK}{\varepsilon}\right)^{\frac{1}{K}}\left(\frac{30}{N^{\frac{1}{2}}}+\frac{32k!}{N}+\frac{8k!}{N^{\frac{1}{2}}}\right)^{\frac{1}{K}}\\
&\leqslant 2\left(d^k_{\frac{\varepsilon}{2k^2K}} 60k!^{1+\varepsilon}\frac{2kK}{\varepsilon}\right)^{\frac{1}{K}} N^{1-\frac{1}{2K}+\varepsilon}\\
&= 2\left(d^k_{\frac{1-2\delta K}{4k^2K^2}} 60k!^{1+\frac{1}{2K}-\delta}\frac{4kK^2}{1-2\delta K}\right)^{\frac{1}{K}} N^{1-\delta}
\end{aligned}$$

for any $\delta < \frac{1}{2K}$, if we take $\varepsilon = \frac{1}{2K} - \delta$. □

2.2. Hua's lemma

Theorem 19 (Hua's lemma)**.** *For $k \geqslant 2$ let $T(\alpha) = \sum_{n=1}^{N} e(\alpha n^k)$. Then*

$$\int_0^1 |T(\alpha)|^{2^k}\, d\alpha \leqslant h_k N^{2^k-k+\varepsilon}, \quad \textit{where} \quad h_k = 2^{2^{k+1}} d^k_{\frac{\varepsilon}{k^2}} k^k.$$

Proof. We will prove by induction on j that

$$\int_0^1 |T(\alpha)|^{2^j}\, d\alpha \leqslant h_j N^{2^j-j+\varepsilon}, \quad \text{where} \quad h_j = 2^{2^{j+1}} d^k_{\frac{\varepsilon}{k^2}} k^j.$$

for $j = 1, \ldots, k$. If $j = 1$, we have

$$\int_0^1 |T(\alpha)|^2\, d\alpha = \int_0^1 T(\alpha)T(-\alpha)d\alpha = \sum_{n=1}^{N}\sum_{m=1}^{N}\int_0^1 e(\alpha(n^k - m^k))d\alpha = N.$$

Let $1 \leqslant j \leqslant k-1$ an assume that it is true for j. Let $f(x) = \alpha x^k$. By Lemma 12 $\Delta_{d_j,\ldots,d_1}(f)(x) = \alpha d_j \cdots d_1 p_{k-j}(x)$, where p_{k-j} is a polynomial of degree $k-j$ with integer coefficients. Applying Lemma 11 with $N_1 = 0$, $N_2 = N$ and $S(f) = T(\alpha)$ we get

$$\begin{aligned}|T(\alpha)|^{2^j} &\leqslant (2N)^{2^j-j-1} \sum_{|d_1|<N} \cdots \sum_{|d_j|<N} \sum_{n\in I(d_j,\dots,d_1)} e(\Delta_{d_j,\dots,d_1}(f)(n)) \\ &\leqslant (2N)^{2^j-j-1} \sum_{|d_1|<N} \cdots \sum_{|d_j|<N} \sum_{n\in I(d_j,\dots,d_1)} e(\alpha d_j \cdots d_1 p_{k-j}(n)),\end{aligned}$$

where $I(d_j, \dots, d_1)$ is an interval of consecutive integers contained in $[1, N]$. Thus

$$|T(\alpha)|^{2^j} \leqslant (2N)^{2^j-j-1} \sum_d r(d)e(\alpha d), \tag{2}$$

where $r(d)$ is the number of choices of $|d_1|, \dots, |d_j| \leqslant N$ and $n \in I(d_j, \dots, d_1)$ such that $d = d_1 \cdots d_j p_{k-j}(n)$. Since the degree of p_{k-j} is $k - j$, it follows that if $d \neq 0$, then by Lemma 13 $|d| \leqslant (j+1)^k N^k \leqslant k^k N^k$ and by Lemma 14 $d(n) \leqslant d_\varepsilon n^\varepsilon$, so

$$r(d) \leqslant d(d)^{j+1}(k-j) \leqslant (d_\varepsilon |d|^\varepsilon)^k k = d_\varepsilon^k |d|^{\varepsilon k} k \leqslant d_\varepsilon^k (kN)^{\varepsilon k^2} k = d_{\frac{\varepsilon}{k^2}}^k (kN)^\varepsilon k.$$

If $d = 0$, we get

$$r(0) \leqslant j(2N)^{j-1}N + (k-j)(2N)^j \leqslant k(2N)^j.$$

On the other hand

$$\begin{aligned}|T(\alpha)|^{2^j} &= T(\alpha)^{2^{j-1}} T(-\alpha)^{2^{j-1}} \\ &= \left(\sum_{x=1}^N e(\alpha x^k)\right)^{2^{j-1}} \left(\sum_{y=1}^N e(\alpha y^k)\right)^{2^{j-1}} \\ &= \sum_{x_1=1}^N \cdots \sum_{x_{2^{j-1}}=1}^N \sum_{y_1=1}^N \cdots \sum_{y_{2^{j-1}}=1}^N e\left(\alpha\left(\sum_{i=1}^{2^{j-1}} x_i^k - \sum_{i=1}^{2^{j-1}} y_i^k\right)\right) \\ &= \sum_d s(d)e(-\alpha d),\end{aligned} \tag{3}$$

where $s(d)$ is the number of representations of d in the form $d = \sum_{i=1}^{2^{j-1}} y_i^k - \sum_{i=1}^{2^{j-1}} x_i^k$ with $1 \leqslant x_i, y_i \leqslant N$. Then

$$\sum_d s(d) = |T(0)|^{2^j} = N^{2^j}.$$

By the inductive hypothesis

$$s(0) = \int_0^1 |T(\alpha)|^{2^j} \, d\alpha \leqslant h_j N^{2^j - j + \varepsilon}.$$

From (2) and (3) follows that

$$\begin{aligned}
\int_0^1 |T(\alpha)|^{2^{j+1}} \, d\alpha &= \int_0^1 |T(\alpha)|^{2^j} \, |T(\alpha)|^{2^j} \, d\alpha \\
&\leqslant (2N)^{2^j - j - 1} \int_0^1 \sum_{d'} r(d') e(\alpha d') \sum_d s(d) e(-\alpha d) d\alpha \\
&= (2N)^{2^j - j - 1} \sum_d r(d) s(d) \\
&= (2N)^{2^j - j - 1} r(0) s(0) + (2N)^{2^j - j - 1} \sum_{d \neq 0} r(d) s(d) \\
&\leqslant (2N)^{2^j - j - 1} k (2N)^j h_j N^{2^j - j + \varepsilon} + (2N)^{2^j - j - 1} d_{\frac{\varepsilon}{k^2}}^k (kN)^\varepsilon k \sum_{d \neq 0} s(d) \\
&\leqslant (2N)^{2^{j+1} - (j+1) + \varepsilon} k h_j + (2N)^{2^j - j - 1} N^\varepsilon N^{2^j} d_{\frac{\varepsilon}{k^2}}^k k^{1+\varepsilon} \\
&\leqslant (2N)^{2^{j+1} - (j+1) + \varepsilon} \left(k h_j + d_{\frac{\varepsilon}{k^2}}^k k^{1+\varepsilon} \right) \\
&= 2^{2^{j+1} - (j+1) + \varepsilon} \left(k 2^{2^{j+1}} d_{\frac{\varepsilon}{k^2}}^k k^j + d_{\frac{\varepsilon}{k^2}}^k k^{1+\varepsilon} \right) N^{2^{j+1} - (j+1) + \varepsilon} \\
&\leqslant 2^{2^{j+2}} d_{\frac{\varepsilon}{k^2}}^k k^{j+1} N^{2^{j+1} - (j+1) + \varepsilon} = h_{j+1} N^{2^{j+1} - (j+1) + \varepsilon}.
\end{aligned}$$

□

2.3. Infinite products

Definition 4. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of complex numbers and let $p_n = \prod_{k=1}^n \alpha_k$ be the nth partial product. We say that the infinite product $\prod_{n=1}^\infty \alpha_n$ converges to $\alpha \neq 0$ if

$$\prod_{n=1}^\infty \alpha_n = \lim_{n \to \infty} p_n = \alpha.$$

Fact 20. *If $\prod_{n=1}^\infty \alpha_n$ converges, then $\lim_{n\to\infty} \alpha_n = 1$.*

Proof. $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{p_n}{p_{n-1}} = 1.$ □

Lemma 21. *Let $a_n \geqslant 0$ for all $n \geqslant 1$. Then $\prod_{n=1}^\infty (1 + a_n)$ converges if and only if $\sum_{n=1}^\infty a_n$ converges.*

Proof. Let $s_n = \sum_{k=1}^n a_k$ and $p_n = \prod_{k=1}^n (1 + a_k)$. Observe that

$$0 \leqslant \sum_{k=1}^{n} a_k < \prod_{k=1}^{n} (1 + a_k) \leqslant \prod_{k=1}^{n} e^{a_k} = e^{\sum_{k=1}^n a_k},$$

that is

$$0 \leqslant s_n < p_n \leqslant e^{s_n}.$$

Since both sequences are increasing, it follows that $\{s_n\}$ converges if and only if $\{p_n\}$ converges. $\square$

Definition 5. We say that $\prod_{n=1}^{\infty}(1 + a_n)$ *converges absolutely* if $\prod_{n=1}^{\infty}(1 + |a_n|)$ converges.

Lemma 22. *If $\prod_{n=1}^{\infty}(1 + a_n)$ converges absolutely, then it converges.*

Proof. Let

$$p_n = \prod_{k=1}^{n}(1 + a_k), \quad P_n = \prod_{k=1}^{n}(1 + |a_k|).$$

The sequence $\{P_n\}$ converges, so the series $\sum_{n=2}^{\infty}(P_n - P_{n-1})$ converges. Observe that

$$\begin{aligned} |p_n - p_{n-1}| = |a_n p_{n-1}| &= \left| a_n \prod_{k=1}^{n-1}(1 + a_k) \right| \\ &\leqslant |a_n| \prod_{k=1}^{n-1}(1 + |a_k|) = |a_n| P_{n-1} = P_n - P_{n-1}. \end{aligned}$$

Therefore $\sum_{n=2}^{\infty} |p_n - p_{n-1}|$ converges, $\sum_{n=2}^{\infty}(p_n - p_{n-1})$ converges and so $\{p_n\}$ converges.

Now we prove that this limit is not zero. Since $\prod_{n=1}^{\infty}(1 + a_n)$ converges absolutely, it follows from Lemma 21 that $\sum_{n=1}^{\infty} |a_n|$ converges and the sequence $\{a_n\}$ converges to zero. Therefore, for all sufficiently large integers n we have $|1 + a_n| \geqslant \frac{1}{2}$ and $\left| -\frac{a_n}{1+a_n} \right| \leqslant 2|a_n|$. It follows that $\sum_{n=1}^{\infty} \left| -\frac{a_n}{1+a_n} \right|$ converges and $\prod_{n=1}^{\infty} \left(1 - \frac{a_n}{1+a_n}\right)$ converges. Thus, the sequence

$$\prod_{k=1}^{n} \left(1 - \frac{a_k}{1 + a_k}\right) = \prod_{k=1}^{n} \frac{1}{1 + a_k} = \frac{1}{\prod_{k=1}^{n}(1 + a_k)} = \frac{1}{p_n}$$

converges to a finite limit and so the limit of the sequence $\{p_n\}$ is nonzero. $\square$

Definition 6. A function f is *multiplicative* if $f(mn) = f(m)f(n)$ for any relatively prime positive integers m and n.

Lemma 23. *Let f be a multiplicative function that is not identically zero. If the series $\sum_{n=1}^{\infty} f(n)$ converges absolutely, then*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p\in\mathbb{P}} \left(1 + \sum_{n=1}^{\infty} f(p^n)\right).$$

Proof. Since $\sum_{n=1}^{\infty} f(n)$ converges absolutely, the series $a_p = \sum_{n=1}^{\infty} f(p^n)$ converges absolutely for every prime p. Also, the series

$$\begin{aligned}\sum_{p\in\mathbb{P}} |a_p| &= \sum_{p\in\mathbb{P}} \left|\sum_{n=1}^{\infty} f(p^n)\right| \\ &\leqslant \sum_{p\in\mathbb{P}} \sum_{n=1}^{\infty} |f(p^n)| \\ &\leqslant \sum_{n=1}^{\infty} |f(n)|\end{aligned}$$

converges, so

$$\prod_{p\in\mathbb{P}} (1 + a_p) = \prod_{p\in\mathbb{P}} \left(1 + \sum_{n=1}^{\infty} f(p^n)\right)$$

converges absolutely and by Lemma 22 it converges.

Let $\varepsilon > 0$ and let N_0 be an integer such that $\sum_{n=N_0}^{\infty} |f(n)| < \varepsilon$. Let $P(n)$ denote the greatest prime factor of n. Let $N \geqslant N_0$. It follows that

$$\prod_{\substack{p\in\mathbb{P}\\ p\leqslant N}} \left(1 + \sum_{n=1}^{\infty} f(p^n)\right) = \sum_{P(n)\leqslant N} f(n)$$

and

$$\left|\sum_{n=1}^{\infty} f(n) - \prod_{\substack{p\in\mathbb{P}\\ p\leqslant N}} \left(1 + \sum_{n=1}^{\infty} f(p^n)\right)\right| = \left|\sum_{n=1}^{\infty} f(n) - \sum_{P(n)\leqslant N} f(n)\right|$$
$$= \left|\sum_{P(n)>N} f(n)\right| \leqslant \sum_{P(n)>N} |f(n)| \leqslant \sum_{n>N} |f(n)| \leqslant \varepsilon.$$

□

3. The circle method

In this section, let $k \geqslant 2$, $s \geqslant 2^k + 1$, $N \geqslant 2^k$, $P = [N^{\frac{1}{k}}]$ and

$$F(\alpha) = \sum_{m=1}^{P} e(\alpha m^k).$$

Then

$$r_{k,s}(N) = \int_0^1 F(\alpha)^s e(-\alpha N) d\alpha.$$

We want to estimate this integral. In order to do this, we will use Hardy's and Littlewood's decomposition of unit interval $[0, 1]$ into major and minor arcs.

Definition 7. Let $0 < \nu < \frac{1}{5}$ and let a and q be integers such that $1 \leqslant q \leqslant P^\nu$, $0 \leqslant a \leqslant q$ and $(a, q) = 1$. Then

$$\mathfrak{M}(q, a) = \left\{ \alpha \in [0, 1] : \left| \alpha - \frac{a}{q} \right| \leqslant \frac{1}{P^{k-\nu}} \right\}$$

is a *major arc* and

$$\mathfrak{M} = \bigcup_{1 \leqslant q \leqslant P^\nu} \bigcup_{\substack{a=0 \\ (a,q)=1}}^{q} \mathfrak{M}(q, a)$$

is the set of major arcs.

$$\mathfrak{m} = [0, 1] \setminus \mathfrak{M}$$

is the set of *minor arcs.*

Lemma 24. *The major arcs are pairwise disjoint.*

Proof. Let $\frac{a}{q} \neq \frac{a'}{q'}$ and suppose that there exists $\alpha \in \mathfrak{M}(q, a) \cap \mathfrak{M}(q', a')$. Then $|aq' - a'q| \geqslant 1$ and

$$\begin{aligned} \frac{1}{P^{2\nu}} &\leqslant \frac{1}{qq'} \\ &\leqslant \left| \frac{a}{q} - \frac{a'}{q'} \right| \\ &\leqslant \left| \alpha - \frac{a}{q} \right| + \left| \alpha - \frac{a'}{q'} \right| \\ &\leqslant \frac{2}{P^{k-\nu}} \end{aligned}$$

which is a contradiction for $P \geqslant 2$ and $k \geqslant 2$. □

Lemma 25. *$\lambda(\mathfrak{M}) \leqslant \frac{2}{P^{k-3\nu}}$, where λ is the Lebesgue measure.*

Proof. $\lambda(\mathfrak{M}(1,0)) = \lambda(\mathfrak{M}(1,1)) = \frac{1}{P^{k-\nu}}$ and for $q \geqslant 2$, $(a,q) = 1$ we have $\lambda(\mathfrak{M}(q,a)) = \frac{2}{P^{k-\nu}}$. Thus

$$\begin{aligned}\lambda(\mathfrak{M}) &= \frac{2}{P^{k-\nu}} + \sum_{2\leqslant q\leqslant P^\nu} \sum_{\substack{a=0\\(a,q)=1}}^{q} \frac{2}{P^{k-\nu}} \leqslant \sum_{1\leqslant q\leqslant P^\nu} \sum_{\substack{a=0\\(a,q)=1}}^{q} \frac{2}{P^{k-\nu}}\\ &= \frac{2}{P^{k-\nu}} \sum_{1\leqslant q\leqslant P^\nu} \varphi(q) \leqslant \frac{2}{P^{k-\nu}} \sum_{1\leqslant q\leqslant P^\nu} q\\ &\leqslant \frac{2}{P^{k-\nu}} \frac{P^\nu(P^\nu+1)}{2} \leqslant \frac{2}{P^{k-3\nu}}.\end{aligned}$$

□

3.1. The minor arcs

Theorem 26. *Let $k \geqslant 2$ and $s \geqslant 2^k + 1$. Then there exists $\delta_1 > 0$ such that*

$$\left|\int_{\mathfrak{m}} F(\alpha)^s e(-N\alpha) d\alpha\right| \leqslant \left(2^K d^k_{\frac{\nu}{4k^2K^2}} 60k!^{1+\frac{\nu}{2K}} \frac{4kK^2}{\nu}\right)^{\frac{s-2^k}{K}} h'_k P^{s-k-\delta_1},$$

where

$$h'_k = 2^{2^{k+1}} d^k_{\frac{\nu}{2Kk^2}} k^k$$

Proof. By Dirichlet's theorem with $Q = P^{k-\nu}$, for each number α there exists a rational number $\frac{a}{q}$ with $1 \leqslant q \leqslant P^{k-\nu}$ and $(a,q) = 1$ such that

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{qP^{k-\nu}} \leqslant \min\left(\frac{1}{P^{k-\nu}}, \frac{1}{q^2}\right).$$

If $\alpha \in \mathfrak{m}$, then $\alpha \notin \mathfrak{M}(1,0) \cup \mathfrak{M}(1,1)$, so

$$\frac{1}{P^{k-\nu}} < \alpha < 1 - \frac{1}{P^{k-\nu}}$$

and $1 \leqslant a \leqslant q-1$. Suppose that $q \leqslant P^\nu$. Then, since $\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{P^{k-\nu}}$, it follows that $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, which is a contradiction. Therefore $P^\nu < q \leqslant P^{k-\nu}$.

Let $K = 2^{k-1}$. By Weyl's inequality with $f(x) = \alpha x^k$, we have

$$\begin{aligned}|F(\alpha)| &\leqslant 2P^{1+\varepsilon} \left(d^k_{\frac{\varepsilon}{2k^2K}} k!^\varepsilon \frac{2kK}{\varepsilon}\right)^{\frac{1}{K}} \left(\frac{30q}{P^k} + \frac{32k!}{P} + \frac{8k!}{q}\right)^{\frac{1}{K}}\\ &\leqslant 2P^{1+\varepsilon} \left(d^k_{\frac{\varepsilon}{2k^2K}} k!^\varepsilon \frac{2kK}{\varepsilon}\right)^{\frac{1}{K}} \left(\frac{30P^{k-\nu}}{P^k} + \frac{32k!}{P} + \frac{8k!}{P^\nu}\right)^{\frac{1}{K}}\\ &\leqslant 2\left(d^k_{\frac{\varepsilon}{2k^2K}} 60k!^{1+\varepsilon} \frac{2kK}{\varepsilon}\right)^{\frac{1}{K}} P^{1+\varepsilon-\frac{\nu}{K}}.\end{aligned}$$

From Hua's lemma we have

$$\begin{aligned}
\left|\int_{\mathfrak{m}} F(\alpha)^s e(-n\alpha)d\alpha\right| &= \left|\int_{\mathfrak{m}} F(\alpha)^{s-2^k} F(\alpha)^{2^k} e(-n\alpha)d\alpha\right| \\
&\leqslant \int_{\mathfrak{m}} |F(\alpha)|^{s-2^k} |F(\alpha)|^{2^k} d\alpha \\
&\leqslant \sup_{\alpha\in\mathfrak{m}} |F(\alpha)|^{s-2^k} \int_0^1 |F(\alpha)|^{2^k} d\alpha \\
&\leqslant \left(2\left(d^k_{\frac{\varepsilon}{2k^2K}} 60k!^{1+\varepsilon} \frac{2kK}{\varepsilon}\right)^{\frac{1}{K}} P^{1+\varepsilon-\frac{\nu}{K}}\right)^{s-2^k} h_k P^{2^k-k+\varepsilon}
\end{aligned}$$

Taking $\varepsilon = \frac{\nu}{2K}$ and $\delta_1 = \frac{\nu(s-2^k)}{K} - (s-2^k+1)\varepsilon > 0$, we get

$$\left|\int_{\mathfrak{m}} F(\alpha)^s e(-n\alpha)d\alpha\right| \leqslant \left(2^K d^k_{\frac{\nu}{4k^2K^2}} 60k!^{1+\frac{\nu}{2K}} \frac{4kK^2}{\nu}\right)^{\frac{s-2^k}{K}} h'_k P^{s-k-\delta_1}.$$

□

3.2. The major arcs

Lemma 27. *Let* $v(\beta) = \sum_{m=1}^{N} \frac{1}{k} m^{\frac{1}{k}-1} e(\beta m)$. *If* $|\beta| \leqslant \frac{1}{2}$, *then*

$$|v(\beta)| \leqslant 4 \min\left(P, |\beta|^{-\frac{1}{k}}\right).$$

Proof. Let $f(x) = \frac{1}{k} x^{\frac{1}{k}-1}$ for $x > 0$. f is positive, decreasing and continuously differentiable. We have

$$|v(\beta)| \leqslant \sum_{m=1}^{N} \frac{1}{k} m^{\frac{1}{k}-1} \leqslant \int_1^N \frac{1}{k} x^{\frac{1}{k}-1} dx + f(1) < N^{\frac{1}{k}} \leqslant 2P.$$

If $|\beta| \leqslant \frac{1}{N}$, then $P \leqslant N^{\frac{1}{k}} \leqslant |\beta|^{-\frac{1}{k}}$ and $|v(\beta)| \leqslant 2\min\left(P, |\beta|^{-\frac{1}{k}}\right)$.

If $\frac{1}{N} < |\beta| \leqslant \frac{1}{2}$, then $|\beta|^{-\frac{1}{k}} < N^{\frac{1}{k}} \leqslant 2P$. Let $M = \left[\frac{1}{|\beta|}\right]$. Then

$$M \leqslant \frac{1}{|\beta|} < M+1 \leqslant N.$$

Let $U(t) = \sum_{1\leqslant n\leqslant t}$. By Lemma 5 we have $|U(t)| \leqslant \frac{1}{2\|\beta\|} = \frac{1}{2|\beta|}$ and by Lemma 1

$$\left|\sum_{m=M+1}^{N} \frac{1}{k} m^{\frac{1}{k}-1}\right| = \left|f(N)U(N) - f(M)U(M) - \int_M^N U(t)f'(t)dt\right|$$
$$\leqslant \frac{1}{2|\beta|}\left(f(N) + f(M) - \int_M^N f'(t)dt\right)$$
$$= \frac{f(M)}{|\beta|} \leqslant \frac{1}{k|\beta|}\left(\frac{1}{2|\beta|}\right)^{\frac{1}{k}-1} \leqslant \frac{1}{|\beta|^{\frac{1}{k}}}.$$

Therefore

$$|v(\beta)| \leqslant \sum_{m=1}^{M} \frac{1}{k} m^{\frac{1}{k}-1} + \sum_{m=M+1}^{N} \frac{1}{k} m^{\frac{1}{k}-1}$$
$$\leqslant M^{\frac{1}{k}} + \frac{1}{|\beta|^{\frac{1}{k}}}$$
$$\leqslant \frac{2}{|\beta|^{\frac{1}{k}}} \leqslant 4 \min\left(P, |\beta|^{-\frac{1}{k}}\right). \qquad \square$$

Lemma 28. *Let a and q be integers such that $1 \leqslant q \leqslant P^\nu$, $0 \leqslant a \leqslant q$ and $(a, q) = 1$. Let*

$$S(q, a) = \sum_{r=1}^{q} e\left(\frac{ar^k}{q}\right).$$

If $\alpha \in \mathfrak{M}(q, a)$, then

$$F(\alpha) = \frac{S(q, a)}{q} v\left(\alpha - \frac{a}{q}\right) \pm 30P^{2\nu},$$

where $\pm x$ denotes any number in the interval $[-x, x]$.

Proof. Let $\beta = \alpha - \frac{a}{q}$. Then $|\beta| \leqslant P^{\nu-k}$ and

$$F(\alpha) - \frac{S(q, a)}{q} v(\beta) = \sum_{m=1}^{P} e(\alpha m^k) - \frac{S(q, a)}{q} \sum_{m=1}^{N} \frac{1}{k} m^{\frac{1}{k}-1} e(\beta m)$$
$$= \sum_{m=1}^{P} e\left(\frac{am^k}{q}\right) e(\beta m^k) - \frac{S(q, a)}{q} \sum_{m=1}^{N} \frac{1}{k} m^{\frac{1}{k}-1} e(\beta m)$$
$$= \sum_{m=1}^{N} u(m) e(\beta m),$$

where

$$u(m) = \begin{cases} e\left(\frac{am}{q}\right) - \frac{S(q,a)}{q}\frac{1}{k}m^{\frac{1}{k}-1} & \text{if } m \text{ is a } k\text{th power} \\ -\frac{S(q,a)}{q}\frac{1}{k}m^{\frac{1}{k}-1} & \text{otherwise.} \end{cases}$$

Let $y \geqslant 1$. Since $|S(q,a)| \leqslant q$, we have

$$\begin{aligned} \sum_{1\leqslant m\leqslant y} e\left(\frac{am^k}{q}\right) &= \sum_{r=1}^{q} e\left(\frac{ar^k}{q}\right) \sum_{\substack{1\leqslant m\leqslant y \\ m\equiv r \pmod q}} 1 \\ &= S(q,a)\left(\frac{y}{q} \pm 1\right) \\ &= y\frac{S(q,a)}{q} \pm q. \end{aligned}$$

Furthermore, for $t \geqslant 1$, we have

$$\begin{aligned} U(t) &= \sum_{1\leqslant m\leqslant t} u(m) \\ &= \sum_{1\leqslant m\leqslant t^{\frac{1}{k}}} e\left(\frac{am^k}{q}\right) - \frac{S(q,a)}{q} \sum_{1\leqslant m\leqslant t} \frac{1}{k}m^{\frac{1}{k}-1} \\ &= t^{\frac{1}{k}}\frac{S(q,a)}{q} \pm q - \frac{S(q,a)}{q}(t^{\frac{1}{k}} \pm 1) = \pm 2q. \end{aligned}$$

Finally, by Lemma 1

$$\begin{aligned} \left|\sum_{m=1}^{N} u(m)e(\beta m)\right| &= \left|e(\beta N)U(N) - 2\pi i\beta \int_1^N e(\beta t)U(t)dt\right| \\ &\leqslant 2q + 4\pi|\beta| q \int_1^N 1dt \\ &\leqslant q(2 + 4\pi|\beta| N) \\ &\leqslant P^\nu(2 + 8\pi P^{\nu-k}P^k) \leqslant 30P^{2\nu}. \end{aligned}$$

□

Theorem 29. *Let*

$$\mathfrak{S}(N,Q) = \sum_{1\leqslant q\leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\frac{S(q,a)}{q}\right)^s e\left(-\frac{Na}{q}\right)$$

and

$$J^*(N) = \int_{-P^{\nu-k}}^{P^{\nu-k}} v(\beta)^s e(-N\beta) d\beta.$$

Then

$$\int_{\mathfrak{M}} F(\alpha)^s e(-N\alpha) d\alpha = \mathfrak{S}(N, P^\nu) J^*(N) \pm 4^{s+2} s P^{s-k-\delta_2},$$

where $\delta_2 = 1 - 5\nu > 0$.

Proof. Let $\alpha \in \mathfrak{M}(q,a)$, $\beta = \alpha - \frac{a}{q}$ and

$$V = V(\alpha, q, a) = \frac{S(q,a)}{q} v\left(\alpha - \frac{a}{q}\right) = \frac{S(q,a)}{q} v(\beta).$$

Since $|S(q,a)| \leqslant q$, by Lemma 27 we have $|V| \leqslant |v(\beta)| \leqslant 4P$. Let $F = F(\alpha)$. Then $|F| \leqslant P$ and $|F - V| \leqslant 30P^{2\nu}$ by Lemma 28. It follows that

$$\begin{aligned} |F^s - V^s| &= |F - V| \left| F^{s-1} + F^{s-2}V + \cdots + FV^{s-2} + V^{s-1} \right| \\ &\leqslant 30P^{2\nu}(4P)^{s-1} s \leqslant 2 \cdot 4^{s+1} s P^{s-1+2\nu}. \end{aligned}$$

Since $\lambda(\mathfrak{M}) \leqslant 2P^{3\nu-k}$ by Lemma 25, it follows that

$$\int_{\mathfrak{M}} |F^s - V^s| \, d\alpha \leqslant 4P^{3\nu-k} 4^{s+1} s P^{s-1+2\nu} = 4^{s+2} s P^{s-k-\delta_2},$$

where $\delta_2 = 1 - 5\nu > 0$. Therefore

$$\begin{aligned} \int_{\mathfrak{M}} F(\alpha)^s e(-N\alpha) d\alpha &= \int_{\mathfrak{M}} V(\alpha, q, a)^s e(-N\alpha) d\alpha \pm 4^{s+2} s P^{s-k-\delta_2} \\ &= \sum_{1 \leqslant q \leqslant P^\nu} \sum_{\substack{a=0 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}(q,a)} V(\alpha, q, a)^s e(-N\alpha) d\alpha \pm 4^{s+2} s P^{s-k-\delta_2}. \end{aligned}$$

If $q \geqslant 2$

$$\begin{aligned} &\int_{\mathfrak{M}(q,a)} V(\alpha, q, a)^s e(-N\alpha) d\alpha \\ &= \int_{\frac{a}{q} - P^{\nu-k}}^{\frac{a}{q} + P^{\nu-k}} V(\alpha, q, a)^s e(-N\alpha) d\alpha \\ &= \int_{-P^{\nu-k}}^{P^{\nu-k}} V\left(\beta + \frac{a}{q}, q, a\right)^s e\left(-N\left(\beta + \frac{a}{q}\right)\right) d\beta \\ &= \left(\frac{S(q,a)}{q}\right)^s e\left(-\frac{Na}{q}\right) \int_{-P^{\nu-k}}^{P^{\nu-k}} v(\beta)^s e(-N\beta) d\beta \\ &= \left(\frac{S(q,a)}{q}\right)^s e\left(-\frac{Na}{q}\right) J^*(N). \end{aligned}$$

If $q=1$, we have $V(\alpha,1,0)=v(\alpha)$ and $V(\alpha,1,1)=v(\alpha-1)$. Therefore

$$\begin{aligned}
&\int_{\mathfrak{M}(1,0)} V(\alpha,q,a)^s e(-N\alpha)d\alpha + \int_{\mathfrak{M}(1,1)} V(\alpha,q,a)^s e(-N\alpha)d\alpha \\
&= \int_0^{P^{\nu-k}} v(\alpha)^s e(-N\alpha)d\alpha + \int_{1-P^{\nu-k}}^1 v(\alpha-1)^s e(-N\alpha)d\alpha \\
&= J^*(N).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_{\mathfrak{M}} F(\alpha)^s e(-N\alpha)d\alpha \\
&= \sum_{2\leqslant q\leqslant P^\nu} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left(\frac{S(q,a)}{q}\right) e\left(-\frac{Na}{q}\right) J^*(N) + J^*(N) \pm 4^{s+2} s P^{s-k-\delta_2} \\
&= \sum_{1\leqslant q\leqslant P^\nu} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left(\frac{S(q,a)}{q}\right) e\left(-\frac{Na}{q}\right) J^*(N) \pm 4^{s+2} s P^{s-k-\delta_2} \\
&= \mathfrak{S}(N,P^\nu) J^*(N) \pm 4^{s+2} s P^{s-k-\delta_2}.
\end{aligned}$$

□

Theorem 30. *Let*

$$J(N) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\beta)^s e(-\beta N)d\beta.$$

There exists $\delta_3>0$ such that $|J(N)|\leqslant 16P^{s-k}$ and $J^*(N)=J(N)\pm 8P^{s-k-\delta_3}$.

Proof. By Lemma 27

$$\begin{aligned}
|J(N)| &\leqslant 8\int_0^{\frac{1}{2}} \min\left(P,|\beta|^{-\frac{1}{k}}\right)^s d\beta \\
&= 8\int_0^{\frac{1}{N}} \min\left(P,|\beta|^{-\frac{1}{k}}\right)^s d\beta + 8\int_{\frac{1}{N}}^{\frac{1}{2}} \min\left(P,|\beta|^{-\frac{1}{k}}\right)^s d\beta \\
&\leqslant 8\int_0^{\frac{1}{N}} P^s d\beta + 8\int_{\frac{1}{N}}^{\frac{1}{2}} \beta^{-\frac{s}{k}} d\beta \\
&\leqslant 8P^{s-k} + 8\frac{\left(\frac{1}{2}\right)^{1-\frac{s}{k}} - \left(\frac{1}{N}\right)^{1-\frac{s}{k}}}{1-\frac{s}{k}} \\
&= 8P^{s-k} + 8k\frac{N^{\frac{s}{k}-1} - 2^{\frac{s}{k}-1}}{s-k} \\
&\leqslant 16P^{s-k}
\end{aligned}$$

and

$$\begin{aligned}
|J(N) - J^*(N)| &= \int_{P^{\nu-k} \leqslant |\beta| \leqslant \frac{1}{2}} v(\beta)^s e(-N\beta) d\beta \\
&\leqslant 2 \int_{P^{\nu-k}}^{\frac{1}{2}} |v(\beta)|^s \, d\beta \\
&\leqslant 8 \int_{P^{\nu-k}}^{\frac{1}{2}} \beta^{-\frac{s}{k}} d\beta \\
&= 8k \frac{\left(P^{k-\nu}\right)^{\frac{s}{k}-1} - 2^{\frac{s}{k}-1}}{s-k} \\
&\leqslant 8P^{s-k-\delta_3},
\end{aligned}$$

where $\delta_3 = v\left(\frac{s}{k} - 1\right) > 0$. □

Lemma 31. *Let α and β be real numbers such that $0 < \beta < 1$ and $\alpha \geqslant \beta$. Then*

$$\sum_{m=1}^{N-1} m^{\beta-1} (N-m)^{\alpha-1} = N^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \pm \frac{2N^{\alpha-1}}{\beta}.$$

Proof. Let $g(x) = x^{\beta-1}(N-x)^{\alpha-1}$. g is positive and differentiable on $(0, N)$ and integrable on $[0, N]$. Moreover, we have

$$\begin{aligned}
\int_0^N g(x)dx &= \int_0^N x^{\beta-1}(N-x)^{\alpha-1}dx \\
&= N^{\alpha+\beta-1} \int_0^1 t^{\beta-1}(1-t)^{\alpha-1}dt \\
&= N^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.
\end{aligned}$$

If $\alpha \geqslant 1$, then

$$g'(x) = g(x)\left(\frac{\beta-1}{x} - \frac{\alpha-1}{N-x}\right) < 0,$$

so g is decreasing on $(0, N)$ and

$$\int_1^N g(x)dx < \sum_{m=1}^{N-1} g(m) < \int_0^{N-1} g(x)dx.$$

Therefore

$$\begin{aligned} 0 &< \int_0^N g(x)dx - \sum_{m=1}^{N-1} g(m) \\ &< \int_0^1 g(x)dx \\ &= \int_0^1 x^{\beta-1}(N-x)^{\alpha-1}dx \\ &\leqslant N^{\alpha-1}\int_0^1 x^{\beta-1}dx = \frac{N^{\alpha-1}}{\beta}. \end{aligned}$$

If $0 < \beta \leqslant \alpha < 1$, then g has a local minimum at

$$c = \frac{(\beta-1)N}{\alpha+\beta-2} \in \left[\tfrac{N}{2}, N\right].$$

This means that g is decreasing for $x \in (0, c)$, therefore

$$\sum_{m=1}^{[c]} g(m) < \int_0^c g(x)dx$$

and

$$\begin{aligned} \sum_{m=1}^{[c]} g(m) &\geqslant \int_1^{[c]} g(x)dx + g([c]) \\ &\geqslant \int_1^c g(x)dx \\ &\geqslant \int_0^c g(x)dx - \frac{N^{\alpha-1}}{\beta}. \end{aligned}$$

If $x \in (c, N)$, then g is increasing, so

$$\sum_{m=[c]+1}^{N-1} g(m) < \int_c^N g(x)dx$$

and

$$\begin{aligned} \sum_{m=[c]+1}^{N-1} g(m) &\geqslant \int_{[c]+1}^{N-1} g(x)dx + g([c]+1) \\ &\geqslant \int_c^{N-1} g(x)dx \\ &\geqslant \int_c^N g(x)dx - \frac{N^{\beta-1}}{\alpha}. \end{aligned}$$

Therefore

$$0 < \int_0^N g(x)dx - \sum_{m=1}^{N-1} g(m) \leqslant \frac{N^{\alpha-1}}{\beta} + \frac{N^{\beta-1}}{\alpha} \leqslant \frac{2N^{\alpha-1}}{\beta}.$$

□

Theorem 32. *If $s \geqslant 2$, then*

$$J(N) = \frac{\Gamma\left(1+\frac{1}{k}\right)^s}{\Gamma\left(\frac{s}{k}\right)} N^{\frac{s}{k}-1} \pm c_s N^{\frac{s-1}{k}-1}, \quad \textit{where} \quad c_s = (5e)^{s-2} \prod_{j=1}^{s-2} \Gamma\left(\tfrac{j}{k}\right).$$

Proof. Let

$$J_s(N) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\beta)^s e(-N\beta) d\beta$$

for $s \geqslant 2$. We will compute this integral by induction on s. Since

$$v(\beta) = \sum_{m=1}^{N} \frac{1}{k} m^{\frac{1}{k}-1} e(\beta m),$$

it follows that

$$v(\beta)^s = \frac{1}{k^s} \sum_{m_1=1}^{N} \cdots \sum_{m_s=1}^{N} (m_1 \cdots m_s)^{\frac{1}{k}-1} e((m_1 + \cdots + m_s)\beta)$$

and

$$\begin{aligned} J_s(N) &= \frac{1}{k^s} \sum_{m_1=1}^{N} \cdots \sum_{m_s=1}^{N} (m_1 \cdots m_s)^{\frac{1}{k}-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} e((m_1 + \cdots + m_s - N)\beta) d\beta \\ &= \frac{1}{k^s} \sum_{\substack{m_1+\cdots+m_s=N \\ 1\leqslant m_i \leqslant N}} (m_1 \cdots m_s)^{\frac{1}{k}-1}. \end{aligned}$$

For $s = 2$, if we apply Lemma 31 with $\alpha = \beta = \frac{1}{k}$, we get

$$\begin{aligned} J_2(N) &= \frac{1}{k^2} \sum_{m=1}^{N-1} m^{\frac{1}{k}-1} (N-m)^{\frac{1}{k}-1} \\ &= \frac{1}{k^2} \frac{\Gamma\left(\frac{1}{k}\right)^2}{\Gamma\left(\frac{2}{k}\right)} N^{\frac{2}{k}-1} \pm \frac{2}{k} N^{\frac{1}{k}-1} \\ &= \frac{\Gamma\left(1+\frac{1}{k}\right)^2}{\Gamma\left(\frac{2}{k}\right)} N^{\frac{2}{k}-1} \pm N^{\frac{1}{k}-1} \end{aligned}$$

as desired.

Let $s \geqslant 2$ and suppose that the theorem holds for s. Then

$$\begin{aligned}
J_{s+1}(N) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\beta)^{s+1} e(-N\beta) d\beta \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\beta) v(\beta)^s e(-N\beta) d\beta \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m=1}^{N} \frac{1}{k} m^{\frac{1}{k}-1} e(\beta m) v(\beta)^s e(-N\beta) d\beta \\
&= \sum_{m=1}^{N} \frac{1}{k} m^{\frac{1}{k}-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\beta)^s e(-(N-m)\beta) d\beta \\
&= \sum_{m=1}^{N} \frac{1}{k} m^{\frac{1}{k}-1} J_s(N-m) \\
&= \frac{\Gamma\left(1+\frac{1}{k}\right)^s}{\Gamma\left(\frac{s}{k}\right)} \sum_{m=1}^{N-1} \frac{1}{k} m^{\frac{1}{k}-1} (N-m)^{\frac{s}{k}-1} \pm \sum_{m=1}^{N} \frac{1}{k} m^{\frac{1}{k}-1} c_s (N-m)^{\frac{s-1}{k}-1}.
\end{aligned}$$

Applying Lemma 31 with $\alpha = \frac{s}{k}$, $\beta = \frac{1}{k}$ to the first term and with $\alpha = \frac{s-1}{k}$, $\beta = \frac{1}{k}$ to the second term, we get

$$\sum_{m=1}^{N-1} \frac{1}{k} m^{\frac{1}{k}-1} (N-m)^{\frac{s}{k}-1} = \frac{1}{k} \frac{\Gamma\left(\frac{s}{k}\right)\Gamma\left(\frac{1}{k}\right)}{\Gamma\left(\frac{s+1}{k}\right)} N^{\frac{s+1}{k}-1} \pm 2N^{\frac{s}{k}-1}$$

and

$$\sum_{m=1}^{N-1} \frac{1}{k} m^{\frac{1}{k}-1} (N-m)^{\frac{s-1}{k}-1} = \frac{1}{k} \frac{\Gamma\left(\frac{s-1}{k}\right)\Gamma\left(\frac{1}{k}\right)}{\Gamma\left(\frac{s}{k}\right)} N^{\frac{s}{k}-1} \pm 2N^{\frac{s-1}{k}-1}$$

Putting it together, we get

$$\begin{aligned}
J_{s+1}(N) &= \frac{\Gamma\left(1+\frac{1}{k}\right)^s}{\Gamma\left(\frac{s}{k}\right)} \left(\frac{1}{k} \frac{\Gamma\left(\frac{s}{k}\right)\Gamma\left(\frac{1}{k}\right)}{\Gamma\left(\frac{s+1}{k}\right)} N^{\frac{s+1}{k}-1} \pm 2N^{\frac{s}{k}-1} \right) \\
&\pm c_s \left(\frac{1}{k} \frac{\Gamma\left(\frac{s-1}{k}\right)\Gamma\left(\frac{1}{k}\right)}{\Gamma\left(\frac{s}{k}\right)} N^{\frac{s}{k}-1} \pm 2N^{\frac{s-1}{k}-1} \right) \\
&= \frac{\Gamma\left(1+\frac{1}{k}\right)^{s+1}}{\Gamma\left(\frac{s+1}{k}\right)} N^{\frac{s+1}{k}-1} \\
&\pm \left(2N^{\frac{s}{k}-1} \frac{\Gamma\left(1+\frac{1}{k}\right)^s}{\Gamma\left(\frac{s}{k}\right)} + \frac{c_s}{k} \frac{\Gamma\left(\frac{s-1}{k}\right)\Gamma\left(\frac{1}{k}\right)}{\Gamma\left(\frac{s}{k}\right)} N^{\frac{s}{k}-1} + 2c_s N^{\frac{s-1}{k}-1} \right)
\end{aligned}$$

Using Lemma 2, we estimate the error term:

$$\begin{aligned}
&2N^{\frac{s}{k}-1}\frac{\Gamma\left(1+\frac{1}{k}\right)^s}{\Gamma\left(\frac{s}{k}\right)}+\frac{c_s}{k}\frac{\Gamma\left(\frac{s-1}{k}\right)\Gamma\left(\frac{1}{k}\right)}{\Gamma\left(\frac{s}{k}\right)}N^{\frac{s}{k}-1}+2c_sN^{\frac{s-1}{k}-1}\\
&\leqslant\left(2\frac{\Gamma\left(1+\frac{1}{k}\right)^s}{\Gamma\left(\frac{s}{k}\right)}+c_s\frac{\Gamma\left(\frac{s-1}{k}\right)\Gamma\left(1+\frac{1}{k}\right)}{\Gamma\left(\frac{s}{k}\right)}+2c_s\right)N^{\frac{s}{k}-1}\\
&\leqslant\left(2e+ec_s\Gamma\left(\tfrac{s-1}{k}\right)+2c_s\right)N^{\frac{s}{k}-1}\\
&\leqslant\left(2ec_s\Gamma\left(\tfrac{s-1}{k}\right)+ec_s\Gamma\left(\tfrac{s-1}{k}\right)+2ec_s\Gamma\left(\tfrac{s-1}{k}\right)\right)N^{\frac{s}{k}-1}\\
&=5ec_s\Gamma\left(\tfrac{s-1}{k}\right)N^{\frac{s}{k}-1}=c_{s+1}N^{\frac{s}{k}-1}.
\end{aligned}$$

□

3.3. The singular series

Definition 8. We define the *singular series* as

$$\mathfrak{S}(N)=\sum_{q=1}^{\infty}A_N(q),$$

where

$$A_N(q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\frac{S(q,a)}{q}\right)^s e\left(\frac{-Na}{q}\right).$$

Lemma 33. *The singular series converges absolutely and uniformly with respect to N.*

Proof. Let $0<\varepsilon<\frac{1}{sK}$. Since $s\geqslant 2^k+1=2K+1$, we have

$$\frac{s}{K}-1-s\varepsilon\geqslant 1+\frac{1}{K}-s\varepsilon=1+\delta_4,$$

where $\delta_4=\frac{1}{K}-s\varepsilon>0$. By Theorem 17

$$|A_N(q)|\leqslant 2^s\left(d^k_{\frac{\varepsilon}{2k^2K}}60k!^{1+\varepsilon}\frac{2kK}{\varepsilon}\right)^{\frac{s}{K}}\frac{q}{q^{\frac{s}{K}-s\varepsilon}}\leqslant 2^s\left(d^k_{\frac{1-\delta_4K}{2k^2K^2s}}60k!^{1+\frac{1-\delta_4K}{Ks}}\frac{2kK^2s}{1-\delta_4K}\right)^{\frac{s}{K}}\frac{1}{q^{1+\delta_4}}.$$

□

Lemma 34. *Let q and r be integers such that $(q,r)=1$. Then*

$$S(qr,ar+bq)=S(q,a)S(r,b).$$

Proof. Since $(q,r)=1$, $\{xr : 1 \leqslant x \leqslant q\} = \{1,\dots,q\}$ and $\{yq : 1 \leqslant y \leqslant r\} = \{1,\dots,r\}$. Every residue modulo qr can be written uniquely as $xr+yq$, where $1 \leqslant x \leqslant q$ and $1 \leqslant y \leqslant r$, so

$$\begin{aligned}
S(qr, ar+bq) &= \sum_{m=1}^{qr} e\left(\frac{(ar+bq)m^k}{qr}\right) \\
&= \sum_{x=1}^{q}\sum_{y=1}^{r} e\left(\frac{(ar+bq)(xr+yq)^k}{qr}\right) \\
&= \sum_{x=1}^{q}\sum_{y=1}^{r} e\left(\frac{(ar+bq)}{qr}\sum_{l=0}^{k}\binom{k}{l}(xr)^l(yq)^{k-l}\right) \\
&= \sum_{x=1}^{q}\sum_{y=1}^{r} e\left(\frac{(ar+bq)}{qr}((xr)^k+(yq)^k)\right) \\
&= \sum_{x=1}^{q}\sum_{y=1}^{r} e\left(\frac{a(xr)^k}{q}\right) e\left(\frac{b(yq)^k}{r}\right) \\
&= \sum_{x=1}^{q} e\left(\frac{ax^k}{q}\right)\sum_{y=1}^{r} e\left(\frac{by^k}{r}\right) \\
&= S(q,a)S(r,b).
\end{aligned}$$

□

Lemma 35. *If $(q,r)=1$, then $A_N(qr)=A_N(q)A_N(r)$.*

Proof. If $(c,qr)=1$, then $c \equiv ar+bq \pmod{q}$, where $(a,q)=(b,r)=1$. From Lemma 34 we have

$$\begin{aligned}
A_N(qr) &= \sum_{\substack{c=1\\(c,qr)=1}}^{qr} \left(\frac{S(qr,c)}{qr}\right)^s e\left(-\frac{cN}{qr}\right) \\
&= \sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{\substack{b=1\\(b,r)=1}}^{r} \left(\frac{S(qr,ar+bq)}{qr}\right)^s e\left(-\frac{(ar+bq)N}{qr}\right) \\
&= \sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{\substack{b=1\\(b,r)=1}}^{r} \left(\frac{S(q,a)}{q}\right)^s \left(\frac{S(r,b)}{r}\right)^s e\left(-\frac{aN}{q}\right) e\left(-\frac{bN}{r}\right) \\
&= \sum_{\substack{a=1\\(a,q)=1}}^{q} \left(\frac{S(q,a)}{q}\right)^s e\left(-\frac{aN}{q}\right) \sum_{\substack{b=1\\(b,r)=1}}^{r} \left(\frac{S(r,b)}{r}\right)^s e\left(-\frac{bN}{r}\right)
\end{aligned}$$

$$= A_N(q)A_N(r).$$

□

Definition 9. For any positive integer q, let $M_N(q)$ be the number of solutions of the congruence

$$x_1^k + \cdots + x_s^k \equiv N \pmod{q},$$

where x_i are integers from the interval $[1, q]$.

Lemma 36. *Let $s \geqslant 2^k + 1$. For every prime p, the series*

$$\chi_N(p) = 1 + \sum_{h=1}^{\infty} A_N(p^h)$$

converges and

$$\chi_N(p) = \lim_{h \to \infty} \frac{M_N(p^h)}{p^{h(s-1)}}.$$

Proof. The convergence of the series follows from Lemma 33. If $(a, q) = d$, then

$$S(q, a) = \sum_{x=1}^{q} e\left(\frac{ax^k}{q}\right) = \sum_{x=1}^{q} e\left(\frac{\frac{a}{d}x^k}{\frac{q}{d}}\right) = d\sum_{x=1}^{\frac{q}{d}} e\left(\frac{\frac{a}{d}x^k}{\frac{q}{d}}\right) = dS\left(\frac{q}{d}, \frac{a}{d}\right).$$

Since

$$\frac{1}{q}\sum_{a=1}^{q} e\left(\frac{am}{q}\right) = \begin{cases} 1 & \text{if } m \equiv 0 \pmod{q} \\ 0 & \text{if } m \not\equiv 0 \pmod{q}, \end{cases}$$

it follows that for any integers $x_1, \ldots, x_s$

$$\frac{1}{q}\sum_{a=1}^{q} e\left(\frac{a(x_1^k + \cdots + x_s^k - N)}{q}\right) = \begin{cases} 1 & \text{if } x_1^k + \cdots + x_s^k \equiv N \pmod{q} \\ 0 & \text{if } x_1^k + \cdots + x_s^k \not\equiv N \pmod{q}, \end{cases}$$

so

$$\begin{aligned}
M_N(q) &= \sum_{x_1=1}^{q} \cdots \sum_{x_s=1}^{q} \frac{1}{q} \sum_{a=1}^{q} e\left(\frac{a(x_1^k + \cdots x_s^k - N)}{q}\right) \\
&= \frac{1}{q} \sum_{a=1}^{q} \sum_{x_1=1}^{q} \cdots \sum_{x_s=1}^{q} e\left(\frac{a(x_1^k + \cdots x_s^k - N)}{q}\right) \\
&= \frac{1}{q} \sum_{a=1}^{q} \sum_{x_1=1}^{q} e\left(\frac{ax_1^k}{q}\right) \cdots \sum_{x_s=1}^{q} e\left(\frac{ax_s^k}{q}\right) e\left(\frac{-aN}{q}\right) \\
&= \frac{1}{q} \sum_{a=1}^{q} S(q,a)^s e\left(\frac{-aN}{q}\right) \\
&= \frac{1}{q} \sum_{d|q} \sum_{\substack{a=1 \\ (a,q)=d}}^{q} S(q,a)^s e\left(\frac{-aN}{q}\right) \\
&= \frac{1}{q} \sum_{d|q} \sum_{\substack{a=1 \\ (a,q)=d}}^{q} d^s S\left(\frac{q}{d}, \frac{a}{d}\right)^s e\left(\frac{-\frac{a}{d}N}{\frac{q}{d}}\right) \\
&= \frac{1}{q} \sum_{d|q} \sum_{\substack{a=1 \\ (a,q)=d}}^{q} q^s \left(\frac{S\left(\frac{q}{d}, \frac{a}{d}\right)}{\frac{q}{d}}\right)^s e\left(\frac{-\frac{a}{d}N}{\frac{q}{d}}\right) \\
&= q^{s-1} \sum_{d|q} A_N\left(\frac{q}{d}\right).
\end{aligned}$$

Therefore

$$\sum_{d|q} A_N\left(\frac{q}{d}\right) = q^{1-s} M_N(q)$$

for $q \geqslant 1$. If we take $q = p^h$, we have

$$1 + \sum_{j=1}^{h} A_N(p^j) = \sum_{d|p^h} A_N\left(\frac{p^h}{d}\right) = p^{h(1-s)} M_N(p^h)$$

and

$$\chi_N(p) = \lim_{h\to\infty}\left(1 + \sum_{j=1}^{h} A_N(p^j)\right) = \lim_{h\to\infty} p^{h(1-s)} M_N(p^h). \qquad \square$$

Lemma 37. *If $s \geqslant 2^k + 1$, then*

$$\mathfrak{S}(N) = \prod_{p\in\mathbb{P}} \chi_N(p).$$

Moreover,

$$\mathfrak{S}(N) \leqslant 2^s \left(d^k_{\frac{1-\delta_4 K}{2k^2K^2s}} 60k!^{1+\frac{1-\delta_4 K}{Ks}} \frac{2kK^2s}{1-\delta_4 K} \right)^{\frac{s}{K}} \left(1 + \frac{1}{\delta_4}\right).$$

for all N.

Proof. From Lemma 33 we know that the series $\sum_{q=1}^{\infty} A_N(q)$ converges absolutely and from Lemma 35 we know that A_N is multiplicative. Thus, Lemma 23 implies that $\mathfrak{S}(N) = \prod_{p\in\mathbb{P}} \chi_N(p)$. Using estimates from the proof of Lemma 33 we get

$$\mathfrak{S}(N) \leqslant c \sum_{q=1}^{\infty} \frac{1}{q^{1+\delta_4}} \leqslant c\left(1 + \int_1^{\infty} \frac{1}{x^{1+\delta_4}} dx\right) = c\left(1 + \frac{1}{\delta_4}\right),$$

where

$$c = 2^s \left(d^k_{\frac{1-\delta_4 K}{2k^2K^2s}} 60k!^{1+\frac{1-\delta_4 K}{Ks}} \frac{2kK^2s}{1-\delta_4 K} \right)^{\frac{s}{K}}.$$

□

Lemma 38. *There exists a prime* $\left(\frac{4c}{\delta_4 \log 2}\right)^{\frac{1}{\delta_4}} \leqslant p_0 \leqslant 2\left(\frac{4c}{\delta_4 \log 2}\right)^{\frac{1}{\delta_4}}$ *such that*

$$\frac{1}{2} \leqslant \prod_{\substack{p\in\mathbb{P} \\ p>p_0}} \chi_N(p)$$

for all N.

Proof. From Lemma 33 we know that

$$|A_N(q)| \leqslant \frac{c}{q^{1+\delta_4}},$$

where

$$c = 2^s \left(d^k_{\frac{1-\delta_4 K}{2k^2K^2s}} 60k!^{1+\frac{1-\delta_4 K}{Ks}} \frac{2kK^2s}{1-\delta_4 K} \right)^{\frac{s}{K}}.$$

Therefore

$$|\chi_N(p) - 1| \leqslant \sum_{j=1}^{\infty} \left|A_N(p^j)\right| \leqslant c \sum_{j=1}^{\infty} \frac{1}{p^{j(1+\delta_4)}} = \frac{c}{p^{1+\delta_4}} \frac{1}{1 - \frac{1}{p^{1+\delta_4}}} \leqslant \frac{2c}{p^{1+\delta_4}}$$

and

$$1 - \frac{2c}{p^{1+\delta_4}} \leqslant \chi_N(p)$$

for all N and p. Now it suffices to find a prime p_0 such that $\frac{1}{2} \leqslant \prod_{\substack{p\in\mathbb{P}\\ p>p_0}} \left(1 - \frac{2c}{p^{1+\delta_4}}\right)$ which, by continuity of logarithm, is equivalent to $\log\frac{1}{2} \leqslant \sum_{\substack{p\in\mathbb{P}\\ p>p_0}} \log\left(1 - \frac{2c}{p^{1+\delta_4}}\right)$. Let us estimate the right-hand side of the last inequality. We will use the fact that $\frac{x}{x+1} < \log(1+x)$ for all $x > -1$. Replacing x with $-\frac{1}{x}$, we get $\frac{1}{1-x} < \log\left(1 - \frac{1}{x}\right)$. Thus

$$\begin{aligned}
0 &\geqslant \sum_{\substack{p\in\mathbb{P}\\ p>p_0}} \log\left(1 - \frac{2c}{p^{1+\delta_4}}\right) \\
&\geqslant \sum_{n=p_0+1}^{\infty} \log\left(1 - \frac{2c}{n^{1+\delta_4}}\right) \\
&\geqslant \int_{p_0}^{\infty} \log\left(1 - \frac{2c}{x^{1+\delta_4}}\right) dx \\
&\geqslant \int_{p_0}^{\infty} \frac{1}{1 - \frac{x^{1+\delta_4}}{2c}} dx \\
&\geqslant -4c \int_{p_0}^{\infty} \frac{1}{x^{1+\delta_4}} dx \\
&= -\frac{4c}{\delta_4 p_0^{\delta_4}} \geqslant \log\frac{1}{2}.
\end{aligned}$$

Solving the last inequality, we get

$$p_0 \geqslant \left(\frac{4c}{\delta_4 \log 2}\right)^{\frac{1}{\delta_4}}.$$

□

Definition 10. Let p be a prime and let $k = p^\tau k_0$, where $\tau \geqslant 0$ and $(p, k_0) = 1$. We define

$$\gamma = \begin{cases} \tau + 1 & \text{if } p > 2 \\ \tau + 2 & \text{if } p = 2. \end{cases}$$

Fact 39. *Let a, b, r be integers. Then $r \equiv 0 \pmod{(a,b)}$ if and only if there exists an integer v such that $av \equiv r \pmod{b}$.*

Proof. Assume that $av \equiv r \pmod{b}$. This means that $b|av - r$, so $(a,b)|av - r$. But $(a,b)|a$, so $r \equiv 0 \pmod{(a,b)}$.

Let $d = (a,b)$ and assume that $r \equiv 0 \pmod{d}$. Let r_0, a_0 and b_0 be integers such that $r = r_0 d$, $a = a_0 d$ and $b = b_0 d$. Then $(a_0, b_0) = 1$. We want to find v such that $b|av - r$. This is equivalent to $b_0|a_0 v - r_0$, which in turn means that there exist integers v and w such that $a_0 v - b_0 w = r_0$. This is true, since a_0 and b_0 are relatively prime. □

Lemma 40. *Let $h \geqslant 3$. Then the subgroup of $\mathbb{Z}^*_{2^h}$ consisting of numbers congruent to 1 modulo 4 is a cyclic group of order 2^{h-2} and 5 is its generator.*

Proof. We want to show that $5^{2^{h-2}} \equiv 1 \pmod{2^h}$ and $5^{2^{h-3}} \not\equiv 1 \pmod{2^h}$, which is equivalent to $2^h | 5^{2^{h-2}} - 1$ and $2^h \nmid 5^{2^{h-3}} - 1$.

$$\begin{aligned} 5^{2^{h-2}} - 1 &= \left(5^{2^{h-3}}\right)^2 - 1 \\ &= \left(5^{2^{h-3}} - 1\right)\left(5^{2^{h-3}} + 1\right) \\ &= \left(5^{2^{h-4}} - 1\right)\left(5^{2^{h-4}} + 1\right)\left(5^{2^{h-3}} + 1\right) \\ &= 4\left(5^{2^0} + 1\right) \cdots \left(5^{2^{h-3}} + 1\right) \end{aligned}$$

Each factor except the first one is congruent to 2 modulo 4 and so the conclusion follows. □

Lemma 41. *Let m be an integer not divisible by p. If the congruence $x^k \equiv m \pmod{p^\gamma}$ is solvable, then the congruence $y^k \equiv m \pmod{p^h}$ is solvable for every $h \geqslant \gamma$.*

Proof. First assume that p is an odd prime. For $h \geqslant \gamma = \tau + 1$, we have

$$(k, \varphi(p^h)) = (k_0 p^\tau, (p-1)p^{h-1}) = (k_0, p-1)p^\tau = (k, \varphi(p^\gamma)).$$

$\mathbb{Z}^*_{p^h}$ is a cyclic group of order $\varphi(p^h) = (p-1)p^h$. Let g be a generator of this group. Let $x^k \equiv m \pmod{p^\gamma}$. Then $(x, p) = 1$ and there exist integers r and u such that $x \equiv g^u \pmod{p^h}$ and $m \equiv g^r \pmod{p^h}$. Since $h \geqslant \gamma$, we have $x \equiv g^u \pmod{p^\gamma}$ and $m \equiv g^r \pmod{p^\gamma}$. and so $ku \equiv r \pmod{\varphi(p^\gamma)}$. By Fact 39 $r \equiv 0 \pmod{(k, \varphi(p^\gamma))}$ and $r \equiv 0 \pmod{(k, \varphi(p^h))}$. Again by Fact 39 there exists an integer v such that $kv \equiv r \pmod{\varphi(p^h)}$. If we let $y = g^v$, then $y^k \equiv m \pmod{p^h}$.

Now assume that $p = 2$. Then m and x are odd. If k is odd, then $\tau = 0$ and $\gamma = 2$. Note that $\{y^k \bmod 2^h : y = 1, 3, \ldots 2^h - 1\} = \{1, 3, \ldots, 2^h - 1\}$, since if $y_1^k \equiv y_2^k \pmod{2^h}$, then $2^h | y_1^k - y_2^k = (y_1 - y_2)(y_1^{k-1} + \cdots + y_2^{k-1})$. Therefore the congruence $y^k \equiv m \pmod{2^h}$ is solvable for all $h \geqslant 1$. If k is even, then $\tau \geqslant 1$, $\gamma \geqslant 3$ and $m \equiv x^k \equiv 1 \pmod 4$. Also, $x^k = (-x)^k$, so we may assume that $x \equiv 1 \pmod 4$. By Lemma 40, we can choose integers r and u such that $m \equiv 5^r \pmod{2^h}$ and $x \equiv 5^u \pmod{2^h}$. Then $x^k \equiv m \pmod{2^\gamma}$ is equivalent to $ku \equiv r \pmod{2^{\gamma-2}}$ and by Fact 39 r is divisible by $(k, 2^{\gamma-2}) = 2^{\gamma-2} = (k, 2^{h-2})$. Again by Fact 39 there exists an integer v such that $kv \equiv r \pmod{2^{h-2}}$. If we let $y = 5^v$, then $y^k \equiv m \pmod{2^h}$. □

Lemma 42. *Let p be prime. If there exist integers $a_1, \ldots, a_s$, not all divisible by p, such that*

$$a_1^k + \cdots + a_s^k \equiv N \ (mod\ p^\gamma),$$

then

$$\chi_N(p) \geqslant \frac{1}{p^{\gamma(s-1)}} > 0.$$

Proof. Suppose that $p \nmid a_1$. Let $h > \gamma$. For each $i = 2, \ldots, s$ there exist $p^{h-\gamma}$ distinct integers $1 \leqslant x_i \leqslant p^h$ such that $x_i \equiv a_i \pmod{p^\gamma}$. Since the congruence

$$x_1^k \equiv N - x_2^k - \cdots - x_s^k \pmod{p^\gamma}$$

is solvable with $x_1 = a_1$, it follows from Lemma 41 that the congruence

$$x_1^k \equiv N - x_2^k - \cdots - x_s^k \pmod{p^h}$$

is also solvable. Thus

$$M_N(p^h) \geqslant p^{(h-\gamma)(s-1)}$$

and by Lemma 36

$$\chi_N(p) = \lim_{h\to\infty} \frac{M_N(p^h)}{p^{h(s-1)}} \geqslant \frac{1}{p^{\gamma(s-1)}} > 0.$$

□

Lemma 43. *If $s \geqslant 2k$ for odd p or $s \geqslant 4k$ for even p, then*

$$\chi_N(p) \geqslant p^{\gamma(1-s)} > 0.$$

Proof. By Lemma 42 it suffices to show that the congruence

$$a_1^k + \cdots + a_s^k \equiv N \pmod{p^\gamma} \tag{4}$$

is solvable in integers a_i not all divisible by p. If N is not divisible by p and the congruence is solvable, then at least one of the integers a_i is not divisible by p. If N is divisible by p, then it suffices to show that the congruence

$$a_1^k + \cdots + a_{s-1}^k + 1^k \equiv N \pmod{p^\gamma}$$

has a solution in integers. This is equivalent to solving

$$a_1^k + \cdots + a_{s-1}^k \equiv N - 1 \pmod{p^\gamma}.$$

In this case $(N-1, p) = 1$. Therefore, it suffices to prove that, for N relatively prime to p, the congruence

$$a_1^k + \cdots + a_s^k \equiv N \pmod{p^\gamma}$$

is solvable in integers with $s \geqslant 2k-1$ if p is odd and with $s \geqslant 4k-1$ if p is even.

Let p be an odd prime and g be a generator of the group $\mathbb{Z}_{p^\gamma}^*$. The order of g is $\varphi(p^\gamma) = (p-1)p^{\gamma-1} = (p-1)p^\tau$. Let $(m,p) = 1$. The integer m is a kth power residue modulo p^γ if and only if there exists an integer x such that $x^k \equiv m \pmod{p^\gamma}$. Let $m \equiv g^r \pmod{p^\gamma}$. Then m is a kth power modulo p^γ if and only if there exists an integer v such that $x \equiv g^v \pmod{p^\gamma}$ and $kv \equiv r \pmod{(p-1)p^\tau}$. Since $k = k_0 p^\tau$ with $(k_0, p) = 1$, it follows from Fact 39 that this congruence is solvable if and only if $r \equiv 0 \pmod{(k_0, p-1)p^\tau}$, so there are

$$\frac{\varphi(p^\gamma)}{(k_0, p-1)p^\tau} = \frac{p-1}{(k_0, p-1)}$$

distinct kth powers modulo p^γ. Let $s(N)$ be the smallest integer s for which the congruence (4) is solvable and let $C(j)$ denote the set of all residues N modulo p^γ relatively prime to p such that $s(N) = j$. If $(m,p) = 1$, then the congruence

$$x_1^k + \cdots x_s^k \equiv N \pmod{p^\gamma}$$

is solvable if and only if the congruence

$$x_1^k + \cdots x_s^k \equiv m^k N \pmod{p^\gamma}$$

is solvable, since we can multiply or divide both sides by m^k. This means that the sets $C(j)$ are closed under multiplication by kth powers, so, if $C(j)$ is non-empty, then $|C(j)| \geqslant \frac{p-1}{(k_0, p-1)}$. Let n be the largest integer such that the set $C(n)$ is non-empty. Let $j < n$ and let N be the smallest integer relatively prime to p such that $s(N) > j$. Since p is an odd prime, it follows that $N - i$ is relatively prime to p for $i = 1$ or 2 and $s(N-i) \leqslant j$. Since $N = (N-1) + 1^k$ and $N = (N-2) + 1^k + 1^k$, it follows that

$$j + 1 \leqslant s(N) \leqslant s(N-i) + 2 \leqslant j + 2$$

and so $s(N-i) = j$ or $j-1$. This implies that no two consecutive sets $C(j)$ are empty for $j = 1, \ldots, n$ and so the number of non-empty sets $C(j)$ is at least $\frac{n+1}{2}$. Since the sets $C(j)$ are pairwise disjoint, it follows that

$$(p-1)p^\tau = \varphi(p^\gamma) = \sum_{\substack{j=1 \\ C(j) \neq \emptyset}}^{n} |C(j)| \geqslant \frac{n+1}{2} \frac{p-1}{(k_0, p-1)}$$

and so

$$n \leqslant 2(k_0, p-1)p^\tau - 1 \leqslant 2k - 1.$$

Therefore, $s(N) \leqslant 2k-1$ if p is an odd prime and N is relatively prime to p.

Now let $p = 2$. If k is odd, then every odd integer is a kth power modulo 2^γ (proved in the proof of Lemma 41), so $s(N) = 1$ for all odd integers N. If k is even, then $k = 2^\tau k_0$ with $\tau \geqslant 1$ and $\gamma = \tau + 2$. We can assume that $1 \leqslant N \leqslant 2^\gamma - 1$. If

$$s = 2^\gamma - 1 = 4 \cdot 2^\tau - 1 \leqslant 4k - 1,$$

then the congruence (4) can be solved by setting $a_i = 1$ for $i = 1, \ldots, N$ and $a_i = 0$ for $i = N+1, \ldots, s$. Therefore, $s(N) \leqslant 4k - 1$ if $p = 2$ and N is odd. □

Theorem 44.

$$c_1 \leqslant \mathfrak{S}(N) \leqslant c_2,$$

where

$$c_1 = \frac{1}{2}\left(4k\left(\frac{4c}{\delta_4 \log 2}\right)^{\frac{1}{\delta_4}}\right)^{2\left(\frac{4c}{\delta_4 \log 2}\right)^{\frac{1}{\delta_4}}(1-s)}$$

and

$$c_2 = 2^s \left(d^k_{\frac{1-\delta_4 K}{2k^2 K^2 s}} 60k!^{1+\frac{1-\delta_4 K}{Ks}} \frac{2kK^2 s}{1 - \delta_4 K}\right)^{\frac{s}{K}} \left(1 + \frac{1}{\delta_4}\right).$$

Moreover,

$$\mathfrak{S}(N, P^\nu) = \mathfrak{S}(N) \pm \frac{c}{\delta_4} P^{-\nu\delta_4}.$$

Proof. From Lemma 37 we have the upper bound. By Lemma 38, there exists a prime $\left(\frac{4c}{\delta_4 \log 2}\right)^{\frac{1}{\delta_4}} \leqslant p_0 \leqslant 2\left(\frac{4c}{\delta_4 \log 2}\right)^{\frac{1}{\delta_4}}$ such that $\frac{1}{2} \leqslant \prod_{\substack{p \in \mathbb{P} \\ p > p_0}} \chi_N(p)$ for all N. Since by Lemma 43

$$\chi_N(p) \geqslant p^{\gamma(1-s)} > 0$$

for all primes p and all N, it follows that

$$\begin{aligned}
\mathfrak{S}(N) = \prod_{p \in \mathbb{P}} \chi_N(p) &\geqslant \frac{1}{2} \prod_{\substack{p \in \mathbb{P} \\ p \leqslant p_0}} \chi_N(p) \\
&\geqslant \frac{1}{2} \prod_{\substack{p \in \mathbb{P} \\ p \leqslant p_0}} p^{\gamma(1-s)} \geqslant \frac{1}{2} \prod_{\substack{p \in \mathbb{P} \\ p \leqslant p_0}} (2kp)^{(1-s)} \\
&\geqslant \frac{1}{2}(2kp_0)^{p_0(1-s)} \\
&\geqslant \frac{1}{2}\left(4k\left(\frac{4c}{\delta_4 \log 2}\right)^{\frac{1}{\delta_4}}\right)^{2\left(\frac{4c}{\delta_4 \log 2}\right)^{\frac{1}{\delta_4}}(1-s)} = c_1 > 0,
\end{aligned}$$

where

$$c = 2^s \left(d^k_{\frac{1-\delta_4 K}{2k^2K^2s}} 60k!^{1+\frac{1-\delta_4 K}{Ks}} \frac{2kK^2s}{1-\delta_4 K} \right)^{\frac{s}{K}}.$$

To prove the last part, note that by Lemma 33, we have

$$|\mathfrak{S}(N) - \mathfrak{S}(N, P^\nu)| \leqslant \sum_{q>P^\nu} |A_N(q)| \leqslant c \sum_{q>P^\nu} \frac{1}{q^{1+\delta_4}} \leqslant c \int_{P^\nu}^\infty \frac{1}{x^{1+\delta_4}} dx = \frac{c}{\delta_4 P^{\nu\delta_4}}$$

□

Theorem 45 (Hardy–Littlewood)**.** *Let $k \geqslant 2$ and $s \geqslant 2^k + 1$. Let $r_{k,s}(N)$ denote the number of representations of N as the sum of s kth powers of positive integers. There exists $\delta > 0$ such that*

$$\begin{aligned} r_{k,s}(N) = {} & \mathfrak{S}(N)\Gamma\left(1+\frac{1}{k}\right)^s \Gamma\left(\frac{s}{k}\right)^{-1} N^{\frac{s}{k}-1} \\ & \pm 2\left(c_s + 8c_2 + \frac{24c}{\delta k} + 4^{s+2}s + mh'_k\right) N^{\frac{s}{k}-1-\delta}, \end{aligned}$$

with

$$c_1 \leqslant \mathfrak{S}(N) \leqslant c_2,$$

where

$$\begin{aligned} m &= \left(2^K d^k_{\frac{\delta}{4kK^2}} 60k!^{1+\frac{1}{10K}} \frac{4K^2}{\delta}\right)^{\frac{s-2^k}{K}} \\ h'_k &= 2^{2^{k+1}} d^k_{\frac{\delta}{2Kk}} k^k \\ c &= 2^s \left(d^k_{\frac{1-\delta K}{2k^2K^2s}} 60k!^{1+\frac{1}{Ks}} \frac{2kK^2}{1-\delta K} s \right)^{\frac{s}{K}} \\ c_1 &= \frac{1}{2}\left(4k\left(\frac{4c}{\delta\log 2}\right)^{\frac{1}{\delta}}\right)^{2\left(\frac{4c}{\delta\log 2}\right)^{\frac{1}{\delta}}(1-s)} \\ c_2 &= c\left(1+\frac{1}{\delta}\right) \\ d_\varepsilon &= \frac{e^{\frac{1}{\varepsilon}} 2^{1+\varepsilon}}{\varepsilon} \\ c_s &= (5e)^{s-2} \prod_{j=1}^{s-2} \Gamma\left(\tfrac{j}{k}\right). \end{aligned}$$

Proof. Let $\delta_0 = \min(1, \delta_1, \delta_2, \delta_3, \nu\delta_4)$ and $\delta = \frac{\delta_0}{k}$. Note that $\delta_0 \leqslant \nu\delta_4 \leqslant \nu$. By Theorem 26, Theorem 29, Theorem 30, Theorem 32, Theorem 44 we have

$$
\begin{aligned}
r_{k,s}(N) &= \int_0^1 F(\alpha)^s e(-N\alpha)d\alpha \\
&= \int_{\mathfrak{M}} F(\alpha)^s e(-N\alpha)d\alpha + \int_{\mathfrak{m}} F(\alpha)^s e(-N\alpha)d\alpha \\
&= \mathfrak{S}(N, P^\nu)J^*(N) \pm 4^{s+2}sP^{s-k-\delta_2} \pm mh'_k P^{s-k-\delta_1} \\
&= \left(\mathfrak{S}(N) \pm \frac{c}{\delta_4}P^{-\nu\delta_4}\right)\left(J(N) \pm 8P^{s-k-\delta_3}\right) \pm 4^{s+2}sP^{s-k-\delta_2} \pm mh'_k P^{s-k-\delta_1} \\
&= \mathfrak{S}(N)J(N) \\
&\quad \pm 8c_2P^{s-k-\delta_3} \pm \frac{16c}{\delta_4}P^{s-k-\nu\delta_4} \pm \frac{8c}{\delta_4}P^{s-k-\delta_3-\nu\delta_4} \\
&\quad \pm 4^{s+2}sP^{s-k-\delta_2} \pm mh'_k P^{s-k-\delta_1} \\
&= \mathfrak{S}(N)J(N) \pm \left(8c_2 + \frac{24c}{\delta_0} + 4^{s+2}s + mh'_k\right) P^{s-k-\delta_0} \\
&= \mathfrak{S}(N)\frac{\Gamma\left(1+\frac{1}{k}\right)^s}{\Gamma\left(\frac{s}{k}\right)} N^{\frac{s}{k}-1} \\
&\quad \pm c_s N^{\frac{s-1}{k}-1} \pm 2\left(8c_2 + \frac{24c}{\delta_0} + 4^{s+2}s + mh'_k\right) N^{\frac{s}{k}-1-\frac{\delta_0}{k}} \\
&= \mathfrak{S}(N)\frac{\Gamma\left(1+\frac{1}{k}\right)^s}{\Gamma\left(\frac{s}{k}\right)} N^{\frac{s}{k}-1} \pm 2\left(c_s + 8c_2 + \frac{24c}{\delta k} + 4^{s+2}s + mh'_k\right) N^{\frac{s}{k}-1-\delta}
\end{aligned}
$$

□

References

[1] M. B. Nathanson, *Additive Number Theory: The Classical Bases*, Springer-Verlag, New York, 1996.