Uniwersytet Wrocławski
Wydział Matematyki i Informatyki
Instytut Matematyczny
specjalność: matematyka teoretyczna

Szymon Cygan

Kondo model of pattern formation

Model Kondo tworzenia się wzorów

Praca magisterska
napisana pod kierunkiem
prof. dr hab. Grzegorza Karcha

Wrocław, 2018

Załącznik Nr 1
do Zasad

....
(imiona i nazwisko)

....
(aktualny adres do korespondencji) (numer PESEL)

....
(adres e-mail)

....
(wydział)

....
(kierunek studiów)

....
(poziom i forma studiów) (numer albumu)

OŚWIADCZENIE O PRAWACH AUTORSKICH I DANYCH OSOBOWYCH

Ja niżej podpisany/a ………………………………………………………………………………….. student/ka Wydziału ………………………………………………………………………………………………. kierunek ……………………………………….. oświadczam, że przedkładana praca dyplomowa na temat:
……
……
………………………………………………………………………………………………..

- jest mojego autorstwa i nie narusza autorskich praw w rozumieniu ustawy z dnia 4 lutego 1994 r. o prawie autorskim i prawach pokrewnych (tekst jednolity: Dz. U. z 2006 r. Nr 90, poz. 631, z późn. zm.) oraz dóbr osobistych chronionych prawem;
- nie zawiera danych i informacji uzyskanych w sposób niedozwolony;
- nie była wcześniej przedmiotem innej urzędowej procedury związanej z nadaniem dyplomu uczelni wyższej lub tytułu zawodowego;
- treść pracy dyplomowej przedstawionej do obrony, zawarta na przekazanym nośniku elektronicznym, jest identyczna z jej wersją drukowaną.

Oświadczam, iż zostałem/am poinformowany/a o prawie dostępu do treści moich danych osobowych oraz ich poprawiania.
Udostępnienie przez mnie danych osobowych ma charakter dobrowolny.

Wyrażam zgodę, na:

- udostępnienie mojej pracy dla celów naukowych i dydaktycznych;
- przetwarzanie moich danych osobowych w myśl ustawy z dnia 29 sierpnia 1997 r. o ochronie danych osobowych (tekst jednolity: Dz. U. z 2014 r., poz. 1182, z późn. zm.);
- umieszczenie mojej pracy w bazie danych Uczelni i jej przechowywanie przez okres stosowny do potrzeb Uczelni;
- wykorzystanie mojej pracy jako elementu komparatywnej bazy danych Uczelni;
- udostępnienie mojej pracy innym podmiotom celem prowadzenia kontroli antyplagiatowej prac dyplomowych i innych tekstów, które zostaną opracowane w przyszłości;
- porównywanie tekstu mojej pracy z tekstami innych prac znajdujących się w bazie porównawczej systemu antyplagiatowego i zasobach Internetu.

Wrocław, …………………………………. ....
(rrrr – mm - dd) (czytelny podpis autora pracy)

Contents

Chapter 1

Introduction

The reaction-diffusion equations can be widely used to describe the behaviour of many biological phenomena like forming patterns in an organism. Japanese biologist Shigeru Kondo presented in his works, [1], [2], [3] a process of forming pattern on a certain species of guppy fish. He observed that two dyes that are present in a fish organism influence each other. That results in forming a diversified patterns on a fish surface. Shigeru Kondo proposed an explanation of this phenomena using an improved reaction-diffusion system. He claimed that the change rate of a dye depends on a dye density and location. It follows in some areas dye change rate depends positively on dye volume and in some areas the dependence is negative. The positive dependence is called activation, and the negative dependence is called inhibition.

1.1 Primary model

Mathematical model proposed by Shigeru Kondo was a differential equation which describes the concentration of a specific substance on a fish skin. Consider bounded and connected subset of plane $\Omega \subset \mathbb{R}^2$. Let u denote the concentration of a substance. The following model was used

$$\frac{\partial u}{\partial t} = (S - a)u \tag{1.1}$$

where $a > 0$ is a constant cell destruction rate and S corresponds to the cell synthesis. Cell synthesis is a process of sophisticated cell interactions, dependent on stimulation operator. Shigeru Kondo in his work claimed that stimulation operator is a convolution with a radial kernel

$$Stim(x, y) = \int \int u(x - \xi, y - \eta) Kernel(\sqrt{\xi^2 + \eta^2}) d\xi d\eta. \tag{1.2}$$

The kernel used in simulations is designed to have positive and negative parts as well. It was observed that without inhibition or activation part no diversified patterns can be achieved and the solution is trivial. In same cases unwanted, high level of cells density can be obtained. To eliminate this problem the saturation function was introduced, to ensure that the density of a substance is tempered. The saturation function was given with the following formula

$$S = \begin{cases} 0, & Stim < 0, \\ Stim, & 0 < Stim \leqslant M, \\ M, & S > M. \end{cases} \tag{1.3}$$

Let K be a convolution kernel and f be a saturation function. Then Kondo model can be rewritten as

$$u_t = -au + f(K * u). \tag{1.4}$$

Shigeru Kondo observed that the model coefficients, destruction rate a, kernel shape, inhibition, activation rate and saturation rate plays the important role in the shape of the obtained pattern. First of all, specific requirements for coefficients needs to be claimed to ensure that the solution will be non-trivial. Secondly, the mutual relations between coefficients determine the properties of the obtained pattern. In some cases the pattern can be symmetric, antisymmetric or connected. Generally, the patterns are regular in some sense. We will explain this phenomena in this work later.

1.2 Equivalent model

The Kondo model presented in (1.4) is difficult to analyse, since it is a convolution differential equation. In this work we will propose an alternative semi reaction - diffusion model, to approximate the convolution model. We will show even the simplified model, is sufficient to prove the existence and stability of non trivial stationary solutions. We introduce the second substance v which will be used to substitute convolution kernel with Laplace equation. Let $\Omega \subset \mathbb{R}^2$ be bounded compact and connected. Let $u, v(x,t) : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ be a functions describing the substances density. Consider the system

$$\begin{aligned} u_t &= -au + f(v), \\ 0 &= cu + (d + \Delta)v, \qquad x \in \Omega, \\ \frac{\partial}{\partial n} v &= 0, \qquad\qquad\qquad x \in \partial\Omega, \\ u(x,0) &= u_0(x), \end{aligned} \tag{1.5}$$

where $a > 0$, c, $d \in \mathbb{R}$, $f \in C^2(\mathbb{R})$. We applied Neumann boundary condition because in Kondo model substance does not diffuse outside of fish organism. We claimed that the substance v does not follow the law of reaction. Observe that our system corresponds to the specific convolution differential equation with kernel K equal to shifted Green kernel. The following formula holds

$$v(x) = c \int_\Omega K(x,y)u(y)dy. \tag{1.6}$$

The value of d determines the core system properties since if d is equal to the eigenvalue of Laplace operator then the second equation can not have solution for every u.

This paper is organized as follows. In chapter 2 we will consider linearised model, with linear function f. We will prove the existence and stability of stationary solutions under specific conditions for model coefficients. We will extend our argument for Laplasian substituted with any operator B such that B^{-1} is compact. In chapter 3 we will prove the existence of non constant stationary solutions for non linear model using Rabinowitz Bifurcation theorem. The stability of solutions will be proven only in dimension 1, since restriction for supremum norm are required. In chapter 4 we will present some numerical simulation to explain the process of forming patterns from random initial conditions. In chapter 5 we will draw a conclusion.

Chapter 2

Linearized model

2.1 Statement of the problem

Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded, connected domain with C^∞ boundary. Consider the linearised system with the Neumann boundary condition

$$\begin{aligned} \frac{\partial u}{\partial t} &= -au + bv \qquad & x \in \Omega, \\ 0 &= cu + dv + \Delta v & \\ \frac{\partial v}{\partial n} &= 0 & x \in \partial\Omega, \\ u_0(x) &= u(x,0). & \end{aligned} \tag{2.1}$$

where $a > 0$ and $b, c, d \in \mathbb{R}$. It is convenient to represent the system (2.1) in matrix form

$$\begin{aligned} \begin{pmatrix} u_t \\ 0 \end{pmatrix} &= \begin{pmatrix} -a & b \\ c & d + \Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad & x \in \Omega, \\ \frac{\partial v}{\partial n} &= 0 & x \in \partial\Omega, \\ u(x,0) &= u_0(x). & \end{aligned} \tag{2.2}$$

Let us define a weak solution of problem (2.1).

Definition 2.1.1. *A pair of functions $u \in C^1([0,\infty), W^{1,2}(\Omega))$, $v \in C^1([0,\infty), W^{1,2}(\Omega))$ is a weak solution of problem* (2.1) *if for every $\phi \in W^{1,2}(\Omega)$ the following equality hold true*

$$\begin{aligned} \int_\Omega \frac{\partial u}{\partial t}\phi &= -\int_\Omega au\phi + \int_\Omega bv\phi, \\ 0 &= \int_\Omega cu\phi + \int_\Omega dv\phi + \int_\Omega \nabla v \nabla \phi. \end{aligned} \tag{2.3}$$

We will prove that if coefficients a, b, c, d fulfils some specific assumptions then there exists a weak solution of (2.1). Moreover, we will prove its stability.

2.2 Laplace operator properties

To prove the existence and stability of solutions we need to recall some basic facts about eigenvalues of Laplacian.

Theorem 2.2.1 (Spectral theorem for Laplace operator with Neumann boundary conditions)**.** *Let $\Omega \subset \mathbb{R}^2$ be an bounded connected domain with C^∞ boundary. The problem*

$$\begin{aligned} \Delta u = \lambda u \qquad & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0 \qquad & x \in \partial\Omega. \end{aligned} \tag{2.4}$$

has countably many eigenvalues which satisfy

$$0 > \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant \cdots \to -\infty$$

and orthonormal basis of eigenfunctions $\{e_j\}_{j=1}^{\infty}$ of $W^{1,2}(\Omega)$ such that

$$\Delta e_j = \lambda_j e_j$$

Proof. The proof can be found in [8, page 234]. □

2.3 Existence of stationary solutions

Theorem 2.3.1. *Assume that $a > 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$. There exists a non zero weak stationary solution of problem (2.1) if and only if*

$$\frac{-ad - bc}{a} = \lambda_k \quad \textit{for some } k \in \mathbb{N}$$

Moreover each such stationary solution is of the form

$$\widetilde{v} = \sum_{l=1}^{n_k} C_l e_{j_l} \quad \textit{and} \quad \widetilde{u} = \frac{b}{a}\widetilde{v},$$

where e_{j_l} are eigenfunctions corresponding to the eigenvalue λ_k, n_k denotes the multiplicity of λ_k eigenvalue and C_l are arbitrary constants.

Proof. A stationary solution of (2.2) fulfils

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -a & b \\ c & d + \Delta \end{pmatrix} \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} \qquad & x \in \Omega, \\ \frac{\partial \widetilde{v}}{n} = 0 \qquad & x \in \partial\Omega. \end{aligned} \tag{2.5}$$

Evaluating $\widetilde{u}$ from the first equation and applying it to the second one yields to the problem

$$\begin{aligned} -\frac{ad + bc}{a}\widetilde{v} = \Delta \widetilde{v} \qquad & x \in \Omega, \\ \frac{\partial \widetilde{v}}{\partial n} = 0 \qquad & x \in \partial\Omega. \end{aligned} \tag{2.6}$$

It follows from Theorem 2.2.1 that system (2.6) has a solution if and only if

$$-\frac{ad + bc}{a} = \lambda_k$$

for some k. There exists n_k eigenfunctions corresponding to λ_k and n_k indices fulfilling

$$-\frac{ad + bc}{a} = \lambda_{j_l} = \lambda_k \quad \text{for each} \quad l \in \{1, 2, \cdots, n_k\}$$

It follows that the solution is of the form $\widetilde{v} = \sum_{l=1}^{n_k} C_l^1 e_{j_l}$ for arbitrary constant C_l^1. Finally,

$$\widetilde{u} = \frac{b}{a}\widetilde{v},$$

which completes the proof. □

2.4 Stability of stationary solutions

We are now ready to formulate and prove a stability theorem for linear model (2.1). We will prove stability for wider assumptions for coefficient a, b, c, d that arises from the Kondo model.

Theorem 2.4.1. *Assume that $a > 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$. Let $(\widetilde{u}, \widetilde{v})$ be a non zero stationary solution of problem (2.1), corresponding to eigenvalue λ_k. The solution $(\widetilde{u}, \widetilde{v})$ is stable if and only if*

$$\frac{ad + bc + a\lambda_j}{d + \lambda_j} \geqslant 0 \quad \text{for each} \quad j \in \mathbb{N},$$

where $\{\lambda_j\}_{j=1}^{\infty}$ are eigenvalues recalled in Theorem 2.2.1.

Proof. From Theorem 2.3.1 the stationary solution is given by

$$(\widetilde{u}, \widetilde{v}) = \left(\frac{b}{a} \sum_{l=1}^{k} C_l^1 e_{j_l}, \sum_{l=1}^{k} C_l^1 e_{j_l} \right). \tag{2.7}$$

Consider the problem with disturbed initial condition $(u_0, v_0) = (\widetilde{u} + u_0', \widetilde{u} + v_0')$. A solution of problem (2.1) can be expressed using orthonormal properties of eigenfunctions

$$\begin{aligned} u(t,x) &= \widetilde{u} + \sum_{j \neq k} a_j(t) e_j(x), \\ v(t,x) &= \widetilde{v} + \sum_{j \neq k} b_j(t) e_j(x). \end{aligned} \tag{2.8}$$

with suitably chosen, a_j and b_j. Substituting those functions into equation (2.1) yields to the fact that it is enough to analyse the stability of the trivial solution $(0,0)$. Hence,

$$\begin{pmatrix} \sum_j a_j'(t) e_j(x) \\ 0 \end{pmatrix} = \begin{pmatrix} -a & b \\ c & d + \Delta \end{pmatrix} \begin{pmatrix} \widetilde{u} + \sum_j a_j(t) e_j(x) \\ \widetilde{v} + \sum_j b_j(t) e_j(x) \end{pmatrix} = \begin{pmatrix} -a & b \\ c & d + \Delta \end{pmatrix} \begin{pmatrix} \sum_j a_j(t) e_j(x) \\ \sum_j b_j(t) e_j(x) \end{pmatrix}. \tag{2.9}$$

Since $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis, the solution of system (2.9) has the following property for each j

$$\begin{pmatrix} a_j'(t) \\ 0 \end{pmatrix} = \begin{pmatrix} -a & b \\ c & d + \lambda_j \end{pmatrix} \begin{pmatrix} a_j(t) \\ b_j(t) \end{pmatrix}. \tag{2.10}$$

Thus, the stability of solutions to problem (2.1) is equivalent to the stability of solution of system (2.10) for each $j \in \mathbb{N}$. Substituting b_j from second equation into the first equation lead us to

$$a_j'(t) = -a a_j(t) - \frac{cd}{d + \lambda_j} a_j = -a_j \frac{ad + bc + a\lambda_j}{d + \lambda_j} = -\mu_j a_j. \tag{2.11}$$

The solution of equation (2.11) is stable if and only if μ_j is non negative for each $j \in \mathbb{N}$. By assumption, the numbers μ_j are well defined and non negative for each j. Moreover $\inf_j \mu_j < 0$, because for each j, $\mu_j \neq 0$ and $\lim_{j \to \infty} \mu_j = -a$. It follows that a_j are dominated decreasing functions and the stationary solution is stable. □

It can be observed that conditions stated in Theorem 2.4.1 determine the range of variability for coefficient d.

Collary 2.4.2. *Denote by $\lambda_{k-}, \lambda_{k+}$ eigenvalues adjacent to λ_k and different from λ_k. If the solution $(\widetilde{u}, \widetilde{v})$ of problem (2.1) corresponding to eigenvalue λ_k, $k > 1$ is stable then*

$$d \in (-\lambda_{k-}, -\lambda_{k+}) \setminus \{-\lambda_k\}.$$

If the stationary solutions corresponds to the first eigenvalue λ_1 then

$$d \in (-\infty, -\lambda_{1+}) \setminus \{-\lambda_1\}.$$

Proof. If $d = -\lambda_j$ for some j, then from equation (2.11) we immediately obtain that the coefficients $(a_j(t), b_j(t)) = (0, 0)$ for any t, which contradicts

$$(a_j(0), b_j(0)) = (< u_0', e_j >, < v_0', e_j >)$$

Moreover, nominators of μ_j forms a increasing divergent sequence and from theorem 2.3.1, the number $\mu_k = 0$. Thus stability is ensured if denominator of μ_j is positive for $j \leqslant k-$ and negative for $j \geqslant k+$. Since, λ_j is monotonic and decreasing, the thesis holds if $\lambda_{k-} < d < \lambda_{k+}$. □

We will now prove the following lemma which will be useful in the subsequent part of this work.

Lemma 2.4.3. *If the assumptions of Theorem 2.4.1 hold, then every stable solution of problem* (2.1) *with initial condition u_0 fulfils*

$$\|u(t)\|_2 \leqslant C^{-\mu t} \|u_0\|_2 \quad \text{for all} \quad t > 0 \tag{2.12}$$

and for some constants $C, \mu > 0$.

Proof. Let us express u_0, v_0 in the orthonormal basis $\{e_j\}$. We obtain

$$\begin{aligned} u_0 &= \sum_j a_j e_j \qquad v_0 = \sum_j b_j e_j, \\ \|u_0\|^2 &= \sum_j a_j^2 \quad \|v_0\|^2 = \sum_j b_j^2. \end{aligned} \tag{2.13}$$

The explicit solution of the linear problem for each a_j follows from ordinary differential equation theory and it is given by formula

$$a_j(t) = a_j e^{-\mu_j t} \tag{2.14}$$

Since e_j is orthonormal basis following inequality holds for each t

$$\|u(t)\|_2 = \sum a_j^2(t) = \sum a_j^2 e^{-2\mu_j t} \leqslant \sum a_j^2 e^{-2 \inf_j \mu_j} = \|u_0\|_2 e^{-2\mu t} \tag{2.15}$$

where $\mu = \inf_{j \in \mathbb{N}} \mu_j$. The number $\inf_{j \in \mathbb{N}} \mu_j$ exists because $\{\mu_j\}_{j \in \mathbb{N}}$ is bounden from below. Moreover □

2.5 Compact operators

The reasoning presented in this chapter can be extended for wilder scope of operators generalizing the Laplace operator. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected, domain with C^∞ boundary. Let $K : W^{1,2}(\Omega) \to W^{1,2}(\Omega)$ be an linear operator, such that K^{-1} is compact, bounded and self-adjoint. Consider the generalized linearised system

$$\begin{aligned} \frac{\partial u}{\partial t} &= -au + bv \qquad\qquad x \in \Omega, \\ 0 &= cu + dv + K(v) \\ u_0(x) &= u(x,0). \end{aligned} \tag{2.16}$$

We will prove the existence and stability of stationary solutions using the same techniques as in the previous section. We need to recall the Hilbert - Schmidt Theorem.

Theorem 2.5.1 (Hilbert - Schmidt spectral theorem)**.** *Let X be Hilbert space. Let $H : X \to X$ be a compact, bounded, self-adjoint operator. Then there exist a finite or infinite sequence of non zero eigenvalues $\{\eta_j\}_j$. Moreover if the sequence is infinite then $\lim_{j\to\infty} \eta_j = 0$. Furthermore there exist an orthonormal basis $\{\varphi_i\}_{i=1}^{\infty}$ of X, such that*

$$H\varphi_j = \eta_j \varphi_j.$$

Proof. Proof can be found in [9, page 728]. □

Remark 2.5.2. *Operator K^{-1} is compact. From Theorem 2.5.1 there exists a sequence of eigenvalues $\lambda_j = \frac{1}{\eta_j}$ of K satisfying $\lim_{j\to\infty} |\lambda_j| = \infty$. The sequence $\{\lambda_j\}$ can be divergent to $\pm\infty$ or can have no limit. It is convenient to index the eigenvalues with integers and order them in an increasing sequence. Observe that, because there is no limit point in set of eigenvalues we have*

$$-\infty \to \lambda_{k-1} \leqslant \lambda_k \leqslant \cdots \leqslant \lambda_{-1} < 0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k \leqslant \lambda_{k+1} \to \infty.$$

Theorem 2.5.3. *Assume that $a > 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$. There exists a non zero weak stationary solution of problem* (2.16) *if and only if*

$$\frac{-ad - bc}{a} = \lambda_k$$

for some k. Moreover each such stationary solution is of the form

$$\widetilde{v} = \sum_{l=1}^{n_k} C_l e_{k_l} \quad \text{and} \quad \widetilde{u} = \frac{b}{a}\widetilde{v},$$

where e_{k_l} are eigenfunctions corresponding to the eigenvalue λ_k, n_k denotes the λ_k multiplicity and C_l are arbitrary constants.

Theorem 2.5.4. *Assume that $a > 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$. Let $(\widetilde{u}, \widetilde{v})$ be a stationary solution of problem 2.16, corresponding to eigenvalue λ_k, $k \in \mathbb{Z}$. The solution $(\widetilde{u}, \widetilde{v})$ is stable if and only if*

$$\frac{ad + bc + a\lambda_j}{d + \lambda_j} \geqslant 0 \quad \text{for each} \quad j \in A \subset \mathbb{Z}$$

where $\{\lambda_j\}_{j\in A\subset\mathbb{Z}}$ are eigenvalues recalled in Collary 2.5.2.

Collary 2.5.5. *If the solution* $(\widetilde{u}, \widetilde{v})$ *of problem* (2.16) *corresponding to eigenvalue* λ_k*, is stable and* λ_k *is neither maximum nor minimum eigenvalue then*

$$d \in (-\lambda_{k-}, -\lambda_{k+}) \setminus \{-\lambda_k\}.$$

If the stationary solutions corresponds to the maximum or minimum eigenvalue then we obtain

- $d \in (-\infty, -\lambda_{k+}) \setminus \{-\lambda_k\}$*, if* λ_k *is minimal eigenvalue*
- $d \in (-\lambda_{k-}, \infty) \setminus \{-\lambda_{k-}\}$*, if* $-\lambda_k$ *is maximal eigenvalue*

where $\lambda_{k-}, \lambda_{k+}$ *denotes eigenvalues adjacent to* λ_k

Lemma 2.5.6. *If the assumptions of Theorem 2.5.4 hold, then every stable solution of problem* (2.16) *with initial condition* u_0 *fulfils*

$$\|u(t)\|_2 \leqslant C^{-\mu t}\|u_0\|_2 \quad \textit{for all} \quad t > 0 \tag{2.17}$$

and for some constants $C, \mu > 0$.

The proofs of Theorem 2.5.3, Theorem 2.5.4, Lemma 2.5.6 and Collary 2.5.5 are completely analogous to the proofs of Theorem 2.3.1, Theorem 2.4.1, Lemma 2.4.3 and Collary 2.4.2, hence we skip it.

Chapter 3

Nonlinear model

Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded, connected set with C^∞ boundary. Consider the nonlinear model

$$\begin{aligned} u_t &= -au + f(v) \qquad && x \in \Omega, \\ 0 &= cu + dv + \Delta v, && \\ \frac{\partial v}{\partial n} &= 0 && x \in \partial\Omega, \\ u_0 &= u(0, x). && \end{aligned} \tag{3.1}$$

For our purpose we will claim that $f(0) = 0$, $f \in C^2(\mathbb{R})$, namely f, f', f" are continuous and bounded. We will also claim that $a > 0$, $c \neq 0$, $d \neq 0$.

3.1 Existence of stationary solution

In this section we will prove the existence of non constant stationary solutions of problem (3.1), which satisfies

$$\begin{aligned} 0 &= -au + f(v) \qquad && x \in \Omega, \\ 0 &= cu + dv + \Delta v, && \\ \frac{\partial v}{\partial n} &= 0 && x \in \partial\Omega. \end{aligned} \tag{3.2}$$

Problem (3.1) can be reduced into one equation by assigning the function $u = \frac{1}{a} f(v)$ from first equation and applying it into the second one. This lead us to the following problem

$$\begin{aligned} 0 &= \frac{c}{a} f(v) + dv + \Delta v \qquad && x \in \Omega, \\ \frac{\partial v}{\partial n} &= 0 && x \in \partial\Omega. \end{aligned} \tag{3.3}$$

Denote $g(v) = \frac{c}{a} f(v)$. We obtain the following system

$$\begin{aligned} 0 &= g(v) + dv + \Delta v \qquad && x \in \Omega, \\ \frac{\partial v}{\partial n} &= 0 && x \in \partial\Omega. \end{aligned} \tag{3.4}$$

Notice that if $\frac{c}{a}$ is positive then g has the same properties as f. Moreover if $d = -\lambda_k$ for some λ_k then the operator $(d + \Delta)^{-1}$ is irreversible, same like in linear model. We will use this fact later. We will now construct a nonlinear functional corresponding to problem (3.4).

3.1.1 Nonlinear functional

Let $G'(u) = g(u)$. Let define us a functional $J : W^{1,2}(\Omega) \to \mathbb{R}$.

$$J(u) = \frac{1}{2}\int |\nabla u|^2 + \int G(u) \tag{3.5}$$

We will now prove the following properties of the functional J.

Lemma 3.1.1. *Functional $J : W^{1,2}(\Omega) \to \mathbb{R}$ is continuous.*

Proof. Let

$$u_1, u_2 \in W^{1,2}(\Omega) \quad \text{and} \quad \|u_1 - u_2\|_{W^{1,2}} \to 0.$$

We will show that $|J(u_1) - J(u_2)| \to 0$.

$$|J(u_1) - J(u_2)| = \frac{1}{2}\int |\nabla u_1|^2 - |\nabla u_1|^2 + \int G(u_1) - G(u_2) \tag{3.6}$$

The convergence of the first integral is determined by the continuity of the norm.

$$\left|\int |\nabla u_1|^2 - |\nabla u_2|^2\right| \leqslant \left|\|u_1\|_{W1,2} - \|u_2\|_{W1,2}\right| \leqslant \|u_1 - u_2\|_{W^{1,2}} \to 0 \tag{3.7}$$

Function G fulfils global Lipschitz condition $\|G(u_1 - u_2)\| < C\|u_1 - u_2\|$. It follows that

$$\int G(u_1) - G(u_2) \leqslant C'\left(\int (G(u_1) - G(u_2))^2\right)^{\frac{1}{2}} \leqslant C" \int \|u_1 - u_2\|_{W^{1,2}} \to 0 \tag{3.8}$$

We obtained that $|J(u_1) - J(u_2)| \to 0$. □

Lemma 3.1.2. *There exists the Gateaux derivative of functional $J : W^{1,2}(\Omega) \to \mathbb{R}$. Moreover $DJ : W^{1,2}(\Omega) \to L(W^{1,2}(\Omega), \mathbb{R})$ is continuous and $J \in C^1(W^{1,2}(\Omega), \mathbb{R})$.*

Proof. Let $v \in W^{1,2}(\Omega)$. The Gateaux derivative of J on the vector v is given by the formula

$$\frac{d}{dt}J(u + tv)\Big|_{t=0} = \int (\nabla u + t\nabla v)\nabla v\Big|_{t=0} + \int g(u + tv)v\Big|_{t=0} = \int \nabla u \nabla v + \int g(u)v. \tag{3.9}$$

We need to show, that if $\|u_1 - u_2\|_{W^{1,2}} \to 0$, then $DJ(u_1) - DJ(u_2)$ tends to 0 in $L(W^{1,2}(\Omega), \mathbb{R})$. Recall that the norm of a functional $l \in L(W^{1,2}(\Omega), \mathbb{R})$ is given by

$$\|l\|_{L(W^{1,2}(\Omega),\mathbb{R})} = \sup_{\substack{x \in W^{1,2}(\Omega) \\ \|x\|=1}} |l(x)| \tag{3.10}$$

It follows that

$$\begin{aligned} \|(DJ(u_1) - DJ(u_2))\|_{L(W^{1,2}(\Omega),\mathbb{R})} &= \sup_{\substack{v \in W^{1,2}(\Omega) \\ \|v\|=1}} \int \nabla(u_1 - u_2)\nabla v + \sup_{\substack{v \in W^{1,2}(\Omega) \\ \|v\|=1}} \int (g(u_1) - g(u_2))v \\ &\leqslant \left(\int (\nabla(u_1 - u_2))^2\right)^{\frac{1}{2}} \sup_{\substack{v \in W^{1,2}(\Omega) \\ \|v\|=1}} \left(\int (\nabla v)^2\right)^{\frac{1}{2}} \\ &+ \left(\int (g(u_1) - g(u_2))^2\right)^{\frac{1}{2}} \sup_{\substack{v \in W^{1,2}(\Omega) \\ \|v\|=1}} \left(\int (\nabla v)^2\right)^{\frac{1}{2}} \\ &\leqslant \|u_1 - u_2\|_{W^{1,2}} + \left(\int (g(u_1) - g(u_2))^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.11}$$

Since g fulfils global Lipschitz condition $\|g(u_1) - g(u_2)\| \leqslant C\|u_1 - u_2\|$. Thus we obtain

$$\left(\int (g(u_1) - g(u_2))^2\right)^{\frac{1}{2}} \leqslant C'\|u_1 - u_2\|_{W^{1,2}} \tag{3.12}$$

and the assumption holds. □

Collary 3.1.3. *There exists the Fréchet derivative of J and its equal to Gateaux derivative*

Proof. This is a well-known result and the proof can be found in [10, page 14] □

Lemma 3.1.4. *There exists the second Gateaux derivative of functional $J : W^{1,2}(\Omega) \to \mathbb{R}$. Moreover $D^2J : W^{1,2}(\Omega) \to L(W^{1,2}(\Omega), L(L(W^{1,2}(\Omega), \mathbb{R}))$ is continuous and $J \in C^2(W^{1,2}(\Omega), \mathbb{R})$.*

Proof. Second derivative is an operator $D^2J : W^{1,2}(\Omega) \to L(W^{1,2}(\Omega), L(W^{1,2}(\Omega), \mathbb{R}))$. Recall that the norm of a functional $l \in L(\Omega, L(\Omega, \mathbb{R}))$ is given by

$$\|l\|_{L(\Omega,L(\Omega,\mathbb{R}))} = \sup_{x\in\Omega} \|l(x)\|_{L(\Omega,\mathbb{R})} = \sup_{x\in\Omega} \sup_{y\in\Omega} |l(x)(y)|, \tag{3.13}$$

where $l(x)(y)$ denotes functional $l(x)$ applied on y. Let $w \in W^{1,2}$. Second derivative is a functional that can be evaluated as follows

$$\begin{aligned}\int \left(D^2J(u)w\right)v &= \int \left(\frac{d}{dt}DJ(u+tw)\Big|_{t=0}\right)v \\ &= \int \frac{d}{dt}\nabla(u+tw)\Big|_{t=0}\nabla v + \int \frac{d}{dt}g(u+tw)\Big|_{t=0} \\ &= \int \nabla w \nabla v + \int g'(u)vw.\end{aligned} \tag{3.14}$$

To show continuity of the second derivative we need to show that if $\|u_1 - u_2\|_{W^{1,2}} \to 0$ then $\|D^2J(u_1) - D^2J(u_2)\|_{L(W^{1,2}(\Omega),L(W^{1,2}(\Omega),\mathbb{R}))} \to 0$.

$$\begin{aligned}\|D^2J(u_1) - D^2J(u_2)\| = &\sup_{\substack{w\in W^{1,2}(\Omega)\\ \|w\|=1|}} \sup_{\substack{v\in W^{1,2}(\Omega)\\ \|v\|=1}} \Bigg|\int \nabla w\nabla v + \int g'(u_1)vw \\ &- \int \nabla w\nabla v - \int g'(u_2)vw\Bigg| \\ \leqslant &\sup_{\substack{w\in W^{1,2}(\Omega)\\ \|w\|=1|}} \sup_{\substack{v\in W^{1,2}(\Omega)\\ \|v\|=1}} \int \left(g'(u_1) - g'(u_2)\right)vw.\end{aligned} \tag{3.15}$$

By the assumption the function g' fulfils the Lipschitz condition, hence from the Schwartz inequality it follows that

$$\begin{aligned}\int \left(f'(u_1) - f'(u_2)\right)vw \leqslant &C"\|u_1 - u_2\|_{W^{1,2}}\left(\int (vw)^2\right)^{\frac{1}{2}} \\ \leqslant &C"\|u_1 - u_2\|_{W^{1,2}}\|v\|_{L^4}\|w\|_{L^4}.\end{aligned}$$

To prove the convergence we need to show that $\|v\|_{L^4}\|w\|_{L^4}$ is bounded. Let us recall Sobolev embedding Theorem 3.2.2. Since the dimension 2 is less then 4, we obtain that $W^{1,2}(\Omega) \subset L^4(\Omega)$ with a continuous embedding and consequently $\|v\|_{L^4}\|w\|_{L^4} \leqslant C\|v\|_{W^{1,2}}\|w\|_{W^{1,2}}$ □

Collary 3.1.5. *There exists the second Fréchet derivative of J and its equal to Gateaux derivative*

Proof. This is a well-known result and the proof can be found in [10, page 14]. □

3.1.2 Existence by the bifurcation theorem

Let X be a real Hilbert space, $\Omega \subseteq X$ be a neighbourhood of 0. Let $L : \Omega \to X$ be a linear continuous operator and let $H \in C(\Omega, X)$. Set $H(u) = o(\|u\|)$ as $u \to 0$. Consider equation

$$Lu + H(u) = \lambda u. \tag{3.16}$$

Obviously, there exists a trivial solution $(\lambda, 0) \in \mathbb{R} \times X$ for each λ.

Definition 3.1.1. *A point $(\mu, 0) \in \mathbb{R} \times X$ is called a bifurcation point for equation* (3.16) *if every neighbourhood of $(\mu, 0)$ contains nontrivial solution of* (3.16).

Lemma 3.1.6. *If $(\mu, 0)$ is a bifurcation point then μ belongs to the spectrum of operator L.*

Proof. Let $(\mu, 0)$ be a bifurcation point for equation (3.16). Consider the family of balls $B_n\left((\mu, 0), \frac{1}{n}\right) \subset \mathbb{R} \times X$. For each n there exists $(\mu_n, u_n) \in B_n$ satisfying

$$Lu_n + H(u_n) = \mu_n u_n \tag{3.17}$$

Obviously, we have $\mu_n \to \mu$. Divide both sides of this equation by the norm of u_n and consider the weak solution

$$\int \frac{u_n}{\|u_n\|} v + \int \frac{H(u_n)}{\|u_n\|} v = \int \mu_n \frac{u_n}{\|u_n\|} v$$

By assumption for H, if $n \to \infty$ we obtain $\int \frac{H(u_n)}{\|u_n\|} v \to 0$. Put $w_n = \frac{u_n}{\|u_n\|}$. Sequence $\{w_n\}_{n=1}^{\infty}$ is bounded, hence it is weakly compact. Let $w_{n_k} \rightharpoonup w$. It follows

$$\int L(w) v = \int \mu w v$$

The equality holds for each $v \in W^{1,2}$, hence μ belongs to the spectrum of operator L.

□

Theorem 3.1.7 (Rabinowitz Bifurcaion Theorem)**.** *Let X be a real Hilbert space, Ω a neighbourhood of 0 in X and $h \in C^2(\Omega, \mathbb{R})$ with $h'(u) = Lu + H(u)$, L being linear and $H(u) = o(\|u\|)$ at $u = 0$. If μ is an isolated eigenvalue of L of finite multiplicity, then $(\mu, 0)$ is a bifurcation point for* (3.16). *Moreover, at least one of the following alternatives occurs:*

1. *$(\mu, 0)$ is and isolated solution of* (3.16) *in $\{\mu\} \times X$*
2. *There is one-sided neighbourhood, Λ of μ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, equation* (3.16) *posses at least two distinct nontrivial solutions.*
3. *There is a neighbourhood I of μ such that for all $\lambda \in I \setminus \{\mu\}$, equation* (3.16) *posses at least one non trivial solution.*

Proof. Proof can be found in [4, page 412]. □

We are now ready to prove the existence of stationary nontrivial solutions for problem (3.4). First, we need to transform the equation 3.4 into the form of

$$Lu + H(u) = \lambda_k u \tag{3.18}$$

for some eigenvalue of Laplacian λ_k recalled in Theorem 2.2.1. Adding λ_k yields to

$$\begin{aligned} &\lambda_k v = \frac{c}{a} f(v) + (d+\lambda_k)v + \Delta v \qquad x \in \Omega, \\ &\frac{\partial v}{\partial n} = 0 \qquad\qquad\qquad\qquad\qquad\quad x \in \partial\Omega. \end{aligned} \tag{3.19}$$

The operators L, H are given by

$$L(v) = \Delta v \quad \text{and} \quad H(v) = \frac{c}{a} f(v) + (d+\lambda_k)v.$$

Let $g = \frac{c}{a} f$ and $G' = g$. Let us define the nonlinear functional $I : W^{1,2}(\Omega) \to \mathbb{R}$

$$I(v) = \frac{1}{2}\int |\nabla v|^2 + \int G(v) + \frac{1}{2}\int (d+\lambda_k)v^2. \tag{3.20}$$

Theorem 3.1.8. *Consider system* (3.4)*. If $f \in C^2(\mathbb{R})$ fulfils the global Lipschitz condition $f(0) = 0$ and $f'(0) = \frac{-a(d+\lambda_k)}{c}$, then there exist a sequence $\{d_n\}_{n=1}^{\infty} \subset \mathbb{R}$, which converges to d and a sequence of nonconstant functions $\{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty} \subset W^{1,2}(\Omega)$, such that $(\widetilde{u_n}, \widetilde{v_n})$ is a stationary solution of*

$$\begin{aligned} &(u)_t = -au + f(v) \qquad x \in \Omega, \\ &0 = cu + d_n v + \Delta v, \\ &\frac{\partial v}{\partial n} = 0 \qquad\qquad\quad x \in \partial\Omega. \end{aligned} \tag{3.21}$$

for each $n \in \mathbb{N}$.

Proof. Let $I(v)$ be as in (3.20) then

$$DI(v) = Lv + H(v).$$

Observe that functional I is a modification of functional J defined in (3.5)

$$I(v) = J(v) + \frac{1}{2}\int (d+\lambda_k)v^2 = J(v) + R(v)$$

From Lemma 3.1.4 we obtain that $J \in C^2(W^{1,2}(\Omega), \mathbb{R})$. Applying Lemma 3.1.4 for the function $f : \mathbb{R} \to \mathbb{R}$, $f(v) = v$ gives $R \in C^2(W^{1,2}(\Omega), \mathbb{R})$. To prove that $H(u) = o(\|v\|)_{W^{1,2}(\Omega)}$, we need to check that if $\|v\|_{W^{1,2}(\Omega)} \to 0$, then

$$\frac{\|H(v)\|_{W^{1,2}}}{\|v\|_{W^{1,2}}} \to 0. \tag{3.22}$$

Since $f(0) = 0$ we obtain

$$\frac{\|\frac{c}{a} f(v) + (d+\lambda_k)v\|_{W^{1,2}}}{\|v\|_{W^{1,2}}} = \frac{\|\frac{c}{a} f(v) - \frac{c}{a} f(0) + (d+\lambda_k)v\|_{W^{1,2}}}{\|v\|_{W^{1,2}}}. \tag{3.23}$$

The right hand side tends to the minus Fréchet derivative of f if $\|v\|_{W^{1,2}} \to 0$. It follows from the assumption that the nominator tends to 0.
Now we can apply Theorem 3.1.7 to obtain that $(\lambda_k, 0)$ is a bifurcation point for system (3.19). It follows that there exists a sequence of $\{\lambda_k^n\}_{n=1}^{\infty}$ convergent to λ_k and a sequence of nonconstant functions $\{v_n\}_{n=1}^{\infty} \subset W^{1,2}(\Omega)$ such that

$$\begin{aligned} &\lambda_k^n v_n = \frac{c}{a} f(v_n) + (d+\lambda_k)v_n + \Delta v_n \qquad x \in \Omega, \\ &\frac{\partial v_n}{\partial n} = 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad x \in \partial\Omega. \end{aligned} \tag{3.24}$$

for each $n \in \mathbb{N}$. Applying

$$d_n = d + \lambda_k - \lambda_k^n \quad \text{and} \quad u_n = -\frac{1}{c}\left(d_n v_n + \Delta b_n\right)$$

yields to the result. □

Collary 3.1.9. *If function f is a linear function equal to $f(v) = bv$ then the condition in 3.1.8 is equal to the condition in theorem 2.3.1.*

3.2 Stability

Let $(\widetilde{u}, \widetilde{v})$ be a stationary nonconstant solution of problem (3.1). Consider the disturbance of stationary solution

$$\begin{aligned} u &= \widetilde{u} + \varphi, \\ v &= \widetilde{v} + \psi. \end{aligned} \tag{3.25}$$

Applying this into the problem (3.1) yields to

$$\begin{aligned} \varphi_t &= -a(\varphi + \widetilde{u}) + f(\widetilde{v} + \psi) \\ 0 &= c\varphi + d\psi + \Delta\psi \qquad && x \in \Omega, \\ \frac{\partial\psi}{\partial n} &= 0 && x \in \partial\Omega. \end{aligned} \tag{3.26}$$

Since, $-a\widetilde{u} = f(\widetilde{v})$, then by Taylor expansion we obtain

$$\begin{aligned} \varphi_t &= -a\varphi + f'(\widetilde{v})\psi + R(\widetilde{u})\varphi^2 \\ 0 &= c\varphi + d\psi + \Delta\psi \qquad && x \in \Omega, \\ \frac{\partial\psi}{\partial n} &= 0 && x \in \partial\Omega. \end{aligned} \tag{3.27}$$

Proposition 3.2.1 (Linearised Stablity)**.** *The stystem* (3.27) *is linearly stable if the system*

$$\begin{aligned} \varphi_t &= -a\varphi + f'(\widetilde{v})\psi \\ 0 &= c\varphi + d\psi + \Delta\psi \qquad && x \in \Omega, \\ \frac{\partial\psi}{\partial n} &= 0 && x \in \partial\Omega. \end{aligned} \tag{3.28}$$

is stable.

We will prove the stability of solutions in case $\Omega \subset \mathbb{R}$. This assumptions is required to ensure, that if the solution is in $W^{1,2}(\Omega)$ then the L^∞ norm is finite. This follows from Sobolev embedding theorem for $p = 2$ and $n = 1$.

Theorem 3.2.2 (Sobolev embedding theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set with boundary of class $C^1(\Omega)$. Let $1 \leqslant p < n$. Then the following embeddings hold*

$W^{1,p}(\Omega) \subset L^{p*}$, *where* $\frac{1}{p*} = \frac{1}{p} - \frac{1}{N}$, *if* $p < n$
$W^{1,p}(\Omega) \subset L^{q}$, *where* $q \in [p, \infty)$, *if* $p = n$
$W^{1,p}(\Omega) \subset L^{\infty}$, *if* $p > n$

Proof. Proof of the theorem can be found in [7], p 285. □

Unfortunately similar relation does not hold in case of $\Omega \subset \mathbb{R}^2$, so we are not able to prove stability in two dimensional case. To achieve required estimate, we need to involve some regularity theory. We are now ready to formulate and prove that solutions achieved by bifurcation theorem are stable under specific conditions.

Theorem 3.2.3. *Let $\Omega = \mathbb{R}$. Let the system $(\widetilde{u}, \widetilde{v})$ be the stationary nonconstant solutions of* (3.1) *obtained through Rabinowitz Bifurcation Theorem. The solution is linearly stable if*

$$\frac{ad + f'(0)c + a\lambda_j}{d + \lambda_j} \geqslant 0 \quad \text{for each} \quad j \in \mathbb{N},$$

where $\{\lambda_j\}_{j=1}^{\infty}$ are eigenvalues recalled in Theorem 2.2.1.

Proof. Let (φ, ψ) be the solution of system

$$\begin{aligned} \varphi_t &= -a\varphi + f'(\widetilde{v})\psi \\ 0 &= c\varphi + d\psi + \Delta\psi \qquad x \in \Omega, \\ \frac{\partial \psi}{\partial n} &= 0 \qquad\qquad\qquad x \in \partial\Omega. \end{aligned} \tag{3.29}$$

and $(\overline{\varphi}, \overline{\psi})$ be the solution of system

$$\begin{aligned} \overline{\varphi}_t &= -a\overline{\varphi} + f'(0)\overline{\psi} \\ 0 &= c\overline{\varphi} + d\overline{\psi} + \Delta\overline{\psi} \qquad x \in \Omega, \\ \frac{\partial \overline{\psi}}{\partial n} &= 0 \qquad\qquad\qquad x \in \partial\Omega. \end{aligned} \tag{3.30}$$

The solution of $(\overline{\varphi}, \overline{\psi})$ is given by the explicit formula in theorem 2.3.1. Define

$$\begin{aligned} \theta &= \varphi - \overline{\varphi}, \\ \omega &= \psi - \overline{\psi}. \end{aligned}$$

We subtract the first equation in 3.30 from the first equation in 3.29 and the second equation in 3.30 from the second equation in 3.29

$$\begin{aligned} \theta_t &= -a\theta + f'(0)\omega + (f'(\widetilde{v}) - f'(0))\psi, \\ 0 &= c\theta + d\omega + \Delta\omega. \end{aligned} \tag{3.31}$$

Evaluating ω from the second equation and applying to the first gives

$$\theta_t = -a\theta - f'(0)c(d + \Delta)^{-1}\theta + (f'(\widetilde{v}) - f'(0))\,(\omega + \overline{\psi}), \tag{3.32}$$

Multiplying by θ and integrating both sides over Ω gives

$$\int_\Omega \theta_t \theta = -a\int_\Omega \theta^2 - f'(0)c\int_\Omega \left((d + \Delta)^{-1}\theta\right)\theta + \int_\Omega (f'(\widetilde{v}) - f'(0))\,(\omega + \overline{\psi})\theta \tag{3.33}$$

The first integral is equal to the time derivative of L^2 norm

$$\int_\Omega \theta_t \theta = \frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}.$$

To show the stability of solutions we need to prove that the integrals on the right hand side are bounded. From lemma 2.4.3, we have

$$\left|-a\int_\Omega \theta^2 - f'(0)c\int_\Omega \left((d+\Delta)^{-1}\theta\right)\theta\right| \leqslant \int_\Omega \left|\left(-a-f'(0)c(d+\Delta)^{-1}\right)\theta\cdot\theta\right| \leqslant -\mu\|\theta\|_{L^2} \tag{3.34}$$

for all $\mu \leqslant 0$. Since $f'(0) \neq \frac{-a(d_n+\lambda_k)}{c}$, coefficient $\mu < 0$. Next, form (3.31) and Cauch - Schwartz inequality we have

$$\begin{aligned}\int_\Omega \left(f'(\widetilde{v})-f'(0)\right)\omega\theta &\leqslant \int_\Omega \left(f'(\widetilde{v})-f'(0)\right)\left(-c(d+\Delta)^{-1}\right)\theta\cdot\theta \\ &\leqslant \|f'(\widetilde{v}-f'(0))\|_{L^\infty}\|\left(-c(d+\Delta)^{-1}\right)\theta\|_{L^2}\|\theta\|_{L^2}\end{aligned}$$

Since, $\|\widetilde{v}\|_{L^\infty}$ is small and f is Lipschitz function then $\|f'(\widetilde{v}-f'(0))\|_{L^\infty} < \varepsilon$. Operator $-c(d+\Delta)^{-1}$ is bounded in L^2, thus we obtain

$$\int_\Omega \left(f'(\widetilde{v})-f'(0)\right)\omega\theta \leqslant \varepsilon C\|\theta\|^2_{L^2}.$$

Last integral can be estimated as follows

$$\int_\Omega \left(f'(\widetilde{v})-f'(0)\right)\overline{\psi}\theta \leqslant \|f'(\widetilde{v}-f'(0))\|_{L^\infty}\|\overline{\psi}\|_{L^2}\|\theta\|_{L^2} \leqslant Ce^{-\mu t}\|\theta\|_{L^2}$$

Finally, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2} &\leqslant -\mu\|\theta\|^2_{L^2} + \varepsilon C\|\theta\|^2_{L^2} + Ce^{-\mu t}\|\theta\|^2_{L^2} \\ &= \|\theta\|^2_{L^2}\left(-\mu+\varepsilon C + Ce^{-\mu t}\right).\end{aligned} \tag{3.35}$$

For sufficiently small ε, the expression $(-\mu+C\varepsilon)$ is negative. It follows that for large t, $\left(-\mu+\varepsilon C+Ce^{-\mu t}\right)$ is negative and less then $\delta < 0$. Therefore $\|\theta\|_{L^2}$ is bounded by $C'e^{-\delta t}$ for sufficiently large t. Since $\|\theta\|_{L^2} > 0$, then $\|\theta\|_{L^2} \xrightarrow{t\to\infty} 0$. □

Chapter 4

Numerical Simulations

4.1 Numerical scheme

It this chapter we will present numerical simulations obtained for the analysed model. Simulations were obtained using following numerical scheme. Consider system of equations

$$\begin{aligned} u_t &= -au + f(v) \qquad && x \in \Omega, \\ 0 &= cu + dv + \Delta v, && \\ \frac{\partial v}{\partial n} &= 0 && x \in \partial\Omega, \\ u_0 &= u(0,x). && \end{aligned} \tag{4.1}$$

Let $\Omega \subset \mathbb{R}^2$ be compact, and connected domain. For our purpose we will claim that Ω is a square domain $\Omega = [0,2\pi] \times [0,2\pi]$. We will claim that constants $a, c, d \in \mathbb{R} \setminus \{0\}$, and $a > 0, b > 0$. The first equation can be approximated using difference quotient with appropriate h. The first component in the difference quotient u_0 is obtained from the system definition. Thus we obtain

$$\begin{aligned} \frac{u_{n+1} - u_n}{h} &= -au_n + f(v_n), \\ u_{n+1} &= (-au_n + f(v_n))\, h + u_n. \end{aligned} \tag{4.2}$$

The function v_n can be evaluated from second equation

$$-cu_n = dv_n + \Delta v_n, \tag{4.3}$$

supplemented with the Neumann boundary conditions. To evaluate the value of v_n we will express the function u_n in the basis of eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$ of Laplacian with Neumann boundary conditions. Let

$$\begin{aligned} u_n(t,x) &= \sum_{j=1}^{\infty} a_j^n(t)\varphi_j(x) \\ a_j &=< u_n, \varphi_j >= \int_\Omega u_n \varphi_j(x) \end{aligned} \tag{4.4}$$

It follows that

$$v_n = \sum_{j=1}^{\infty} a_j^n(t) \frac{-c}{d + \lambda_j} \varphi_j(x). \tag{4.5}$$

To ensure numerical efficiency we will approximate v_n with finite sum of order N

$$v_n = \sum_{j=1}^{N} a_j^n(t) \frac{-c}{d + \lambda_j} \varphi_j(x). \tag{4.6}$$

The eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$ for quadratic domain are given by explicit formula

$$\{\varphi_j\}_{j=1}^{\infty} = \{cos\left(\frac{nx}{2}\right) cos\left(\frac{my}{2}\right), \quad m, n \in \mathbb{N}\}_{m=1,n=1}^{\infty}. \tag{4.7}$$

To facilitate further analysis we will change the indexing of eigenfunctions such that

$$\varphi_{n,m}(x, y) = cos\left(\frac{nx}{2}\right) cos\left(\frac{my}{2}\right)$$

It follows immediately that the eigenvalue $\lambda_{n,m}$ corresponding to $\varphi_{n,m}$ is equal to

$$\lambda_{n,m} = -\frac{n^2 + m^2}{4}.$$

Notice that $\lambda_{m,n} = \lambda_{n,m}$. The dot product over quadratic domain is equal to the double integral

$$< u_n, \varphi_{n,m} >= \int_0^{2\pi} \int_0^{2\pi} u_n(x, y) cos\left(\frac{nx}{2}\right) cos\left(\frac{my}{2}\right) dxdy$$

In our numerical simulations integrating over domain is approximated by finite sums over homogeneous domain partition.

4.2 Simulation results

Parameters that were used in numerical simulations are the compromise between required computation accuracy and accepted calculation time. Parameters used in simulation are following Grid size = 50 x 50, Step size h = 0.1, Number of eigenfunctions N = 11.

4.2.1 Linear model

Results presented in this section were obtained for linear model, with function $f(v) = bv$. Simulation results are shown of Fig 4.1, Fig 4.2, Fig 4.3 and Fig 4.4. The pattern of solution u is presented in the first row. The coefficients of eigenfunctions $\varphi_{m,n}$ are shown in the second row. Each column shows the solution in time $t = 0,\ 3,\ 6,\ 9$. Function u_0 is chosen randomly with two dimensional uniform distribution. System coefficients are fixed and $a = 1,\ b = 0.2,\ c = -1$. Coefficient d is chosen to fulfil

$$a\lambda_{m,n} + ad + bc = 0$$

for some $\lambda_{m,n}$. It can be seen that all eigenfunctions which do not corresponds to eigenvalue $\lambda_{m,n}$ converges to zero. Notice that for $d = 2.2$ and $d = 4.7$ the stationary solution is symmetric. Those values corresponds to the single eigenvalues $\lambda_{1,1}$ and $\lambda_{1.5,1.5}$ respectively. It implies that there exists only one eigenfunction for each eigenvalue which is symmetric, so the solution is symmetric as well. If $d = 4.45$ or $d = 8.7$ then there exists two eigenfunctions for each eigenvalue which do not converge to 0. Those eigenfunctions are $\varphi_{2,0.5}, \varphi_{0.5,2}$ and $\varphi_{2.5,1.5}, \varphi_{1.5,2.5}$ for $d = 4.45$ and $d = 8.7$ respectively. The coefficients for each eigenvalue are determined by the initial condition which is random. It follows that the stationary solution is not symmetric.

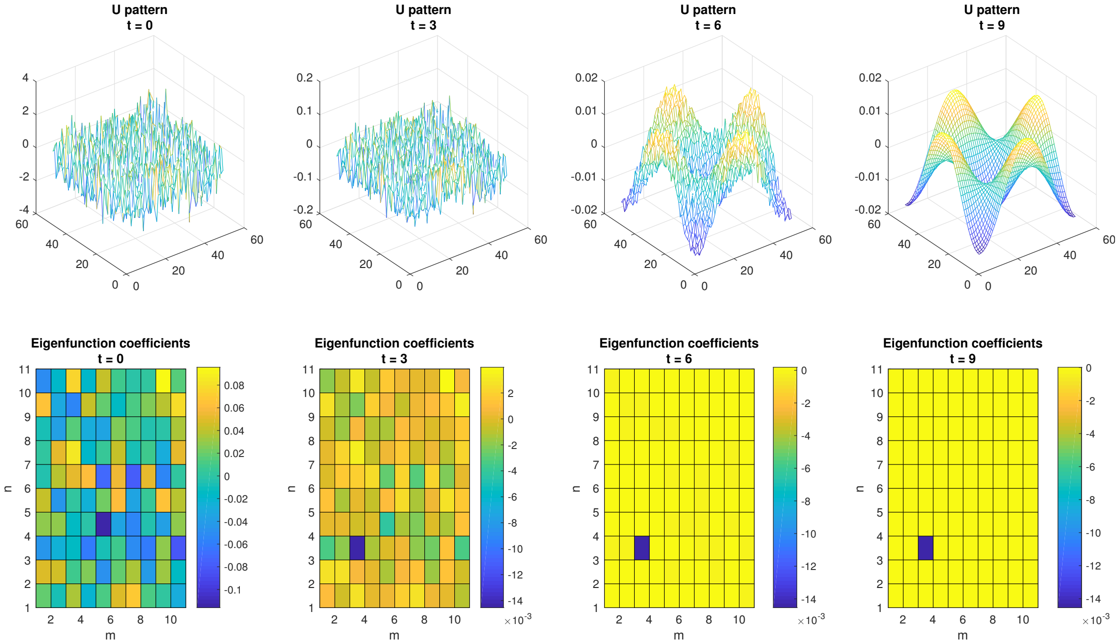

Figure 4.1: Patterns and eigenfunctions coefficients for $t = 0, 3, 6, 9$. Parameter value: $d = 2.2$

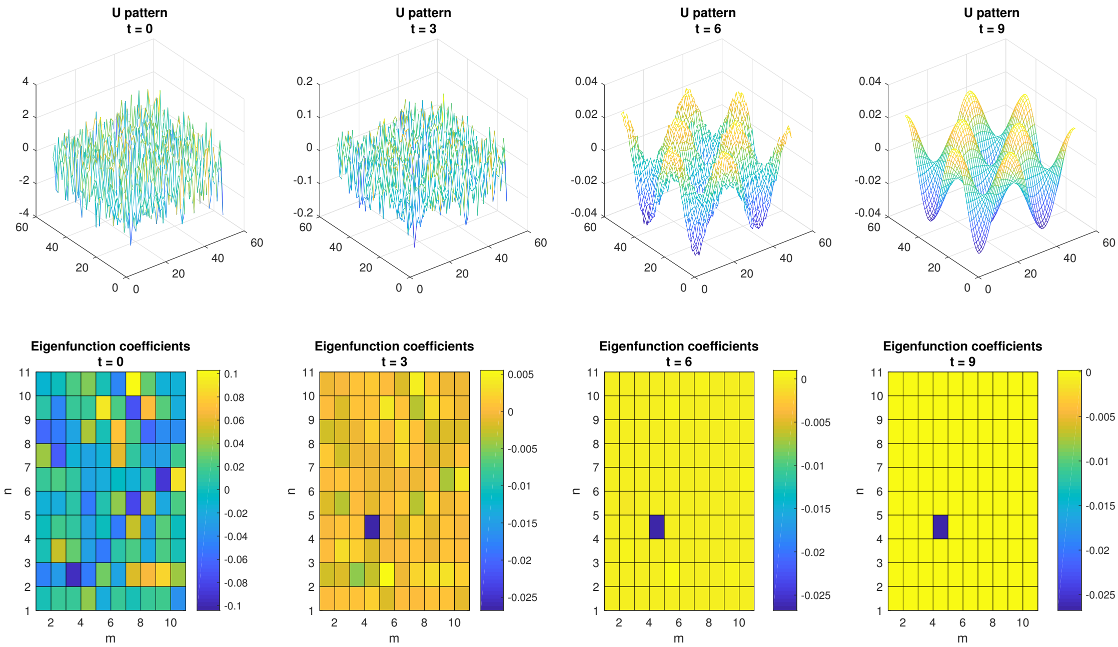

Figure 4.2: Patterns and eigenfunctions coefficients for $t = 0, 3, 6, 9$. Parameter value: $d = 4.7$

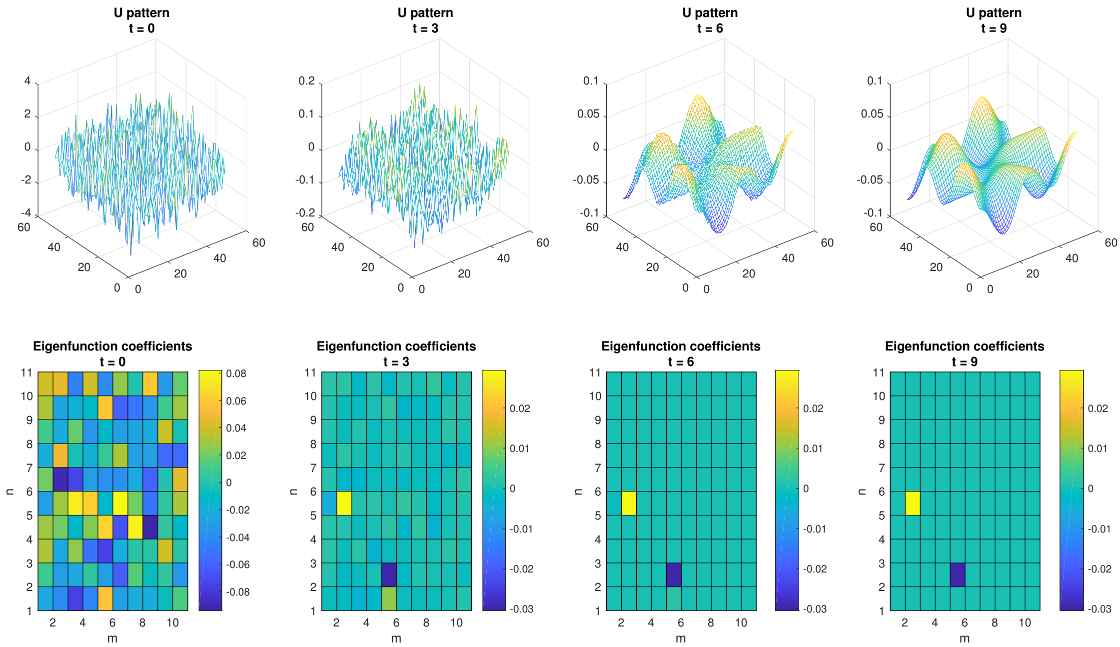

Figure 4.3: Patterns and eigenfunctions coefficients for $t = 0, 3, 6, 9$. Parameter value: $d = 4.45$

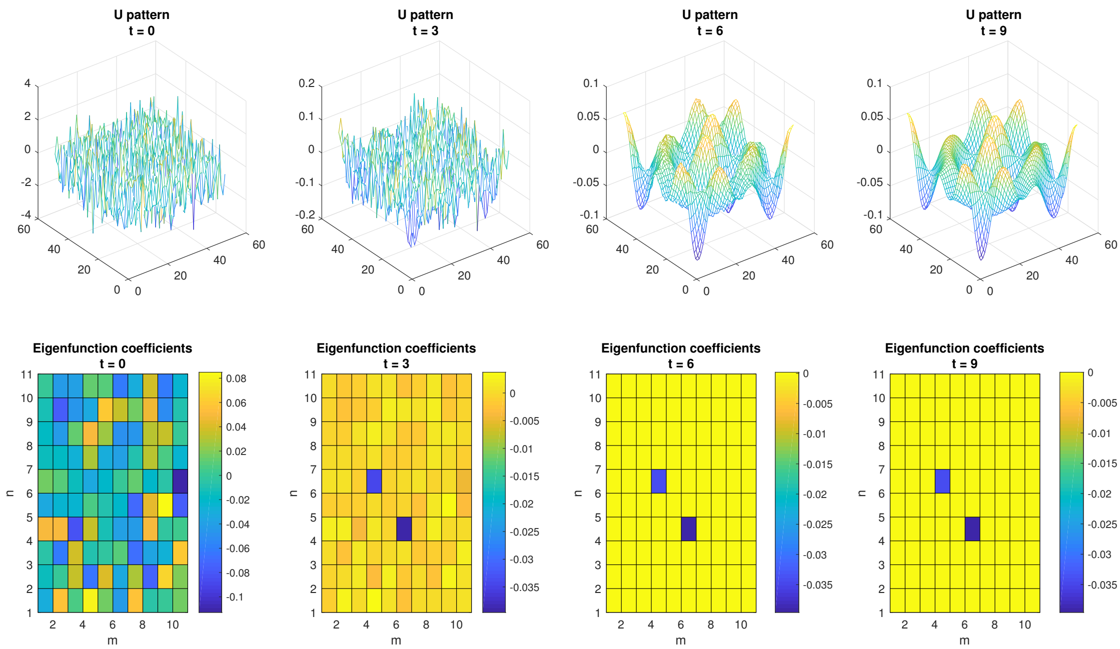

Figure 4.4: Patterns and eigenfunctions coefficients for $t = 0, 3, 6, 9$. Parameter value: $d = 8.7$

4.2.2 Nonlinear model

Results presented in this section were obtained for nonlinear model (4.1), with saturation function $f : \mathbb{R} \to \mathbb{R}$ equal to

$$f(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leqslant x \leqslant x_m, \\ x_m, & x > x_m. \end{cases} \tag{4.8}$$

Coefficients a is the same as in linear model. Modification of a coefficient is not required to achieve all possible patterns. Variation of a determines the convergence rate of eigenvalue coefficients, however required patterns can be achieved by appropriate modifications of remaining parameters. Parameters c was set to 30. Larger values of c ensures that more eigenfunctions becomes unstable, which results in more complicated and diversified patterns. The value of parameter x_m is set to 1. It was observed that the values of lower and upper saturation limits does not change solution pattern, but results only in rescaling and shifting of the obtained solution. Various patterns can be achieved by modifying d coefficients, since it determines the range of unstable eigenfunctions.

Simulation results are presented in Fig. 4.7, Fig 4.6, Fig 4.5. Notice that the stabilization time of the solution is noticeably shorter than in linear case. This phenomena occurs, since in linear case all but one eigenfunctions were decreasing, and the remaining one stayed constant. In nonlinear case some eigenfunctions are unstable, and this empowers the relative difference between stable and unstable eigenfunctions. Observe that patterns on Fig. 4.5 and Fig. 4.7 are regular in some sense. Patterns obtained at level 1, are similar to patterns obtained at level 0. Pattern in the Fig. 4.6 is not regular. Pattern at level 0 is significantly different then pattern at level 1. The crucial difference between those patterns is that in case 1 and 3, there exist a few eigenvalues which are significantly larger then the rest of eigenfunctions.

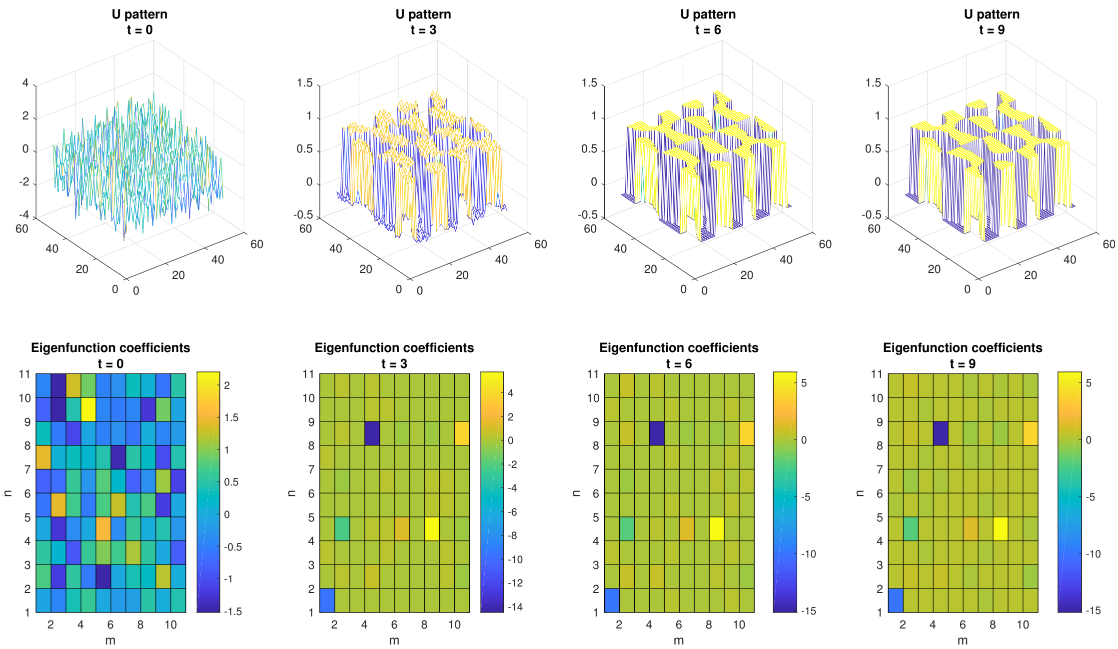

Figure 4.5: Patterns and eigenfunctions coefficients for $t = 0, 3, 6, 9$. Parameter value: $d = 14.45$

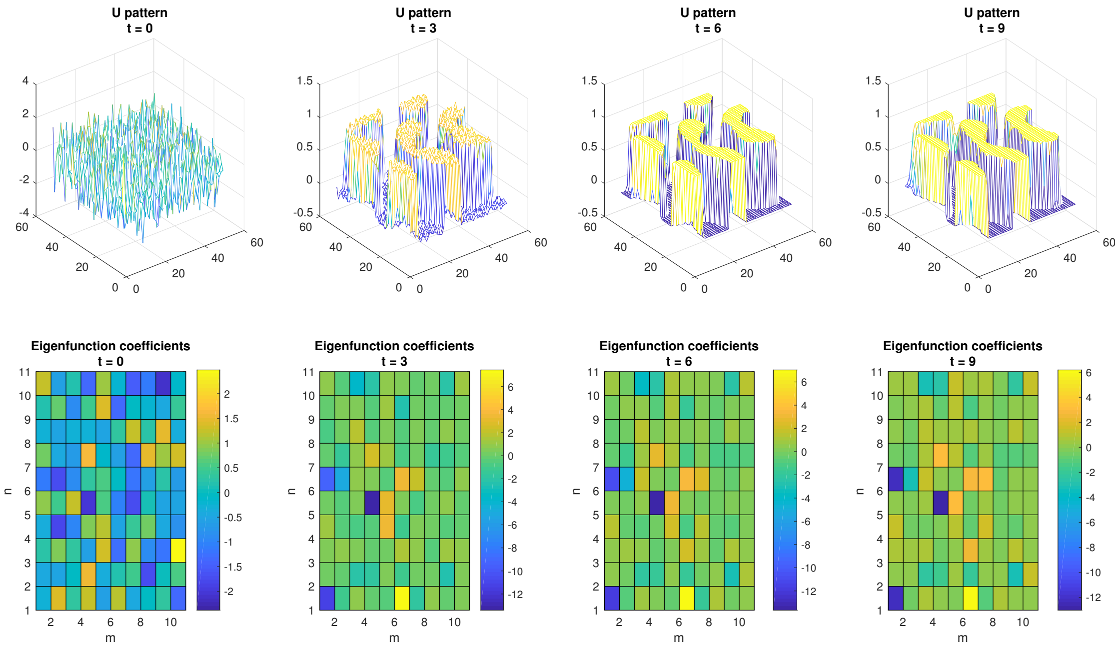

Figure 4.6: Patterns and eigenfunctions coefficients for $t = 0, 3, 6, 9$. Parameter value: $d = 5.22$

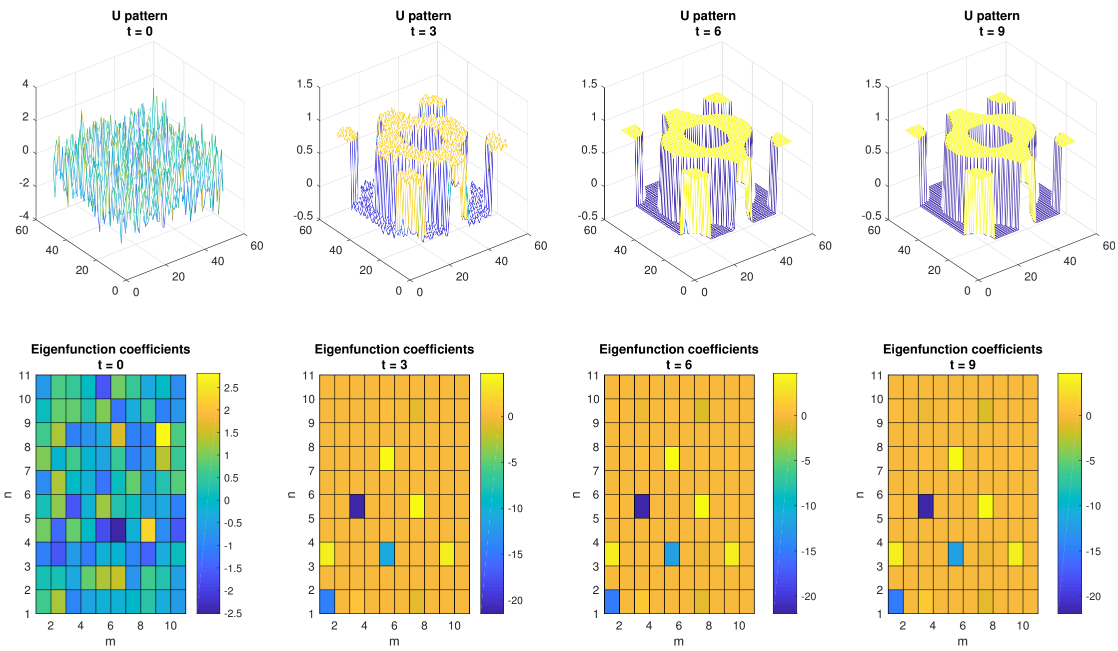

Figure 4.7: Patterns and eigenfunctions coefficients for $t = 0, 3, 6, 9$. Parameter value: $d = 4.98$

Chapter 5

Conclusion

In this work we have attempted to explain the process of forming pattern obtained by Shigeru Kondo in numerical simulations. We proposed an alternative model semi reaction-diffusion model which is significantly simpler to analyse than convolution differential system proposed by Shigeru Kondo. Our model corresponds to the specific kernel in the Kondo model, with kernel K equal to the Green kernel. Moreover, by substituting the Laplace operator with other compact operator we can extend our result for wider class of convolution kernels. Using base Laplace operator properties we have proven that on certain conditions, non constant stable stationary solutions may exist. The shape of the obtained pattern strongly depends on the coefficients value and initial conditions.

In the linear case we have proven that the stable stationary solution is a linear combination of eigenfunctions. The eigenvalue corresponding to those eigenfunctions fulfils specified condition for model coefficients. The value of eigenfunctions are determined by the initial condition. Notice that we did not use the fact that the dimension is equal to 2. All of those theorems holds for any $\mathbb{R}^n$. Results from this section are perfectly confirmed by numerical simulations. Eigenfunctions that do not correspond to the distinguished eigenvalue are vanishing exponentially.

In the non linear case we have proven the existence using Rabinowitz Bifurcation Theorem. We have obtained, that under specific assumptions for derivative of f there may exist a set of coefficients d_n such that bifurcation occurs and non constant stationary solution exists. However, proposed theorem is insufficient to decide whether there exists stationary solution for a given d. It follows from the properties of involved theorem. We obtain only the discontinuous branch of bifurcation points. To obtain the complete theorem for any d we would need to involve the bifurcation theorem with continuous branch of bifurcation points. Notice that, in Kondo model saturation function does not follow the C^2 assumption. To tackle with this problem, one can approximate polyline with appropriate smooth function.

Numerical simulations for non linear model demonstrates significantly different mechanism of forming stationary solutions. Obtained stationary solutions are not small. It can be observed that for specific valued of f', c, d there exists a set of eigenfunctions, that are unstable. As time passes, those eigenfunctions are growing exponentially, until first eigenfunction reaches saturation level. Then the solution concentrates near the values of u_{max} and u_{min} and subsequently it stabilizes. Those patterns match the patterns obtained by Shigeru Kondo. It shows, that there exists different mechanism of pattern formation that need to be investigated.

To prove the stability of non constant solutions, we required the L^∞ estimates for function u. Unfortunately, we are not able to ensure appropriate estimates in dimension larger then 1. To submit proper proof for $n = 2$ we would need to involve relevant theorems from regularity theory.

Bibliography

[1] A reaction-diffusion wave on the skin of the marine angelfish Pomacanthus, S. Kondo, R. Asai, Nature vol 376, 31 August, 1995

[2] S. Kondo, M. Iwashita, M. Yamaguchi, How animals get their skin patterns: fish pigment pattern as a live Turing wave, Int. J. Dev. Biol. 53: 851-856 (2009)

[3] S. Kondo, An updated kernel-based Turing model for studying the mechanisms of biological pattern formation, Journal of Theoretical Biology 414 (2017) 120–127

[4] P. H. Rabinowitz, A bifurcation for Potential Operators, Journal of functional analysis 25, 412-424 (1977)

[5] P. Drábek, J. Milota, Methods of Nonlinear Analysis. Applications to Differential Equations, ISBN 978-3-7643-8146-2, 2007 Birkhäuser Verlag AG

[6] M. Reed, B. Simon, Methods of modern mathematical physics: Functional Analysis, ISBN 0-12-585050-6, 1980 California

[7] H. Brezis, Functional analysis, Sobolev Spaces and Partial Differential Equations, ISBN 978-0-387-70913-0

[8] J. Jost, Partial Differential Equations ISBN 0-387-95428-7

[9] L. C. Evans, Partial Differential Equations, ISBN 978-0-8218-4974-3

[10] A. Ambrosetti, G. Prodi, A Primer of Nonlinear Analysis, ISBN 0-521-37390-5